Schreiber *alias* Grammateus:
From "false position" to the tentative beginnings of algebra

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Today information circulates at the speed of light and rapid progress is made in all domains. That sometimes makes us forget that in former times everything was infinitely slower: both the transmission of knowledge, but also its development.

Some argue that algebra "has been around since 825" because the oldest treaty that speaks of it – written in Arabic in Baghdad by Al-Khwarizmi – dates from that period. But what they don't think about is when it arrived on our shores (its transmission), nor the form it took during the Middle Ages (its development).

That is why we feel it is interesting to 1) explore how algebra was presented in Germany in 1521, by one of the first authors to write about it, and 2) show that it was not yet considered the mathematical *sine qua non*. Indeed, it was rivalled by the far more ancient technique of false position.

REGULA FALSI: BETWEEN MATHEMATICS AND MUSIC/ACCOUNTANCY

The chapter title page (below) announces the start of the section on false position and algebra. Running to 78 pages, the section is preceded by 95 pages of classical arithmetic, with a strong emphasis placed on the Rule of Three (cross-multiplication). It is followed by 78 pages about music, accountancy and the calibration of barrels. Far from being simply anecdotal or aesthetically pleasing, this page is most intriguing to those who take an interest in the history of mathematics.

The first thing one notices is that the size of the script clearly foregrounds false position – called *regula falsi* ("false rule" in Latin) – compared to algebra, which was called *Coss* in Germany at the time (from *cosa*, "thing", which meant "unknown" in Italian). This shows that in 1521, algebra was by no means the standard technique when it came to solving problems. Its diminutive status compared to false position was due to the fact that it was not yet fully effective, as we will see below.

The page contains another paradox. In a book devoted to new, written calculation – a development made possible by the numerals presented on its first few pages, and which the author presents as neither Indian nor Arabic – the engraving shows people performing calculations … using counting tokens and abaci! And this engraving is reproduced three times throughout the book! That is a sign that in Germany in 1521, the new calculation methods were still very much in the minority.

FALSE POSITION

This now largely forgotten calculatory method was very well known in the Middle Ages: the panel provides a brief summary for the modern-day reader.[1]

1. See also Gavin & Schärlig, *Longtemps avant l'algèbre : La Fausse Position, ou comment on a posé le faux pour connaître le vrai...,* Presses Polytechniques et Universitaires Romandes, Lausanne, 2012, 222 p.

Simple false position

The false position method, which was already known in Ancient Egypt, is used to solve linear problems (i.e. involving direction proportion). It involves *assuming* an easily computed – but deliberately *incorrect* – value for the unknown number. The mathematician carries out the proof, but obviously does not – generally – obtain the correct answer. He then turns to the rule of three, using the incorrect value, the incorrect answer, and the number he wishes to obtain. And thus he arrives at the correct answer.

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An example. A man has found some treasure. He has squandered half of it, replaced a quarter of it, then squandered a further eighth. He has 50 gold coins left. What was the original treasure? For the sake of practicality, the mathematician assumes 8. Halved, the treasure equals 4. Replenished by a quarter, it increases to 6. Reduced by one eighth, it is worth just 5. This is one tenth of the 50 gold coins. So 8 is ten times too small. The answer is 80.

It is worth noting in passing that good old maths problems involving running taps and unplugged bathtubs – bad memories for many a school-leaver – are much easier to solve using false position than with algebra. And that comes as no surprise: they were devised a very long time ago to illustrate how false position works and were only later swept up into the corpus of algebra lessons, where they are rather out of place.

Here is another example, from 14th-century Byzantium. A container is equipped with five tubes: the first one fills it in 2 hours, the second in 3 hours, and the third in 4 hours. But, at the same time, one tube empties it in 6 hours, and another in 4 hours. When all tubes are working at the same time, how long does it take to fill the container?

The mathematician assumes 12 hours. The number of times the container is filled is 6 + 4 + 3 – 2 – 3 = 8. If it is filled 8 times in 12 hours, it will be filled once in 1½ hours. The answer is therefore one and a half hours. This would be a lot more difficult to work out using algebra!

Another possible false position: the mathematician assumes 1 hour. The container has been filled at a rate of 1/2 + 1/3 + 1/4 (feed tubes) and emptied at a rate of 1/6 + 1/4 (drainage tubes). It is therefore two-thirds full. It will be completely full in 3/2 x 1 hour, i.e. in 1½ hours.

There is another *regula falsi* method: double false position. This is what the author uses here, even though he does not say so. Since his description of the method is a little dense for us – though undoubtedly pedagogical for its day – let's briefly explain what it is in modern mathematical symbolism. The mathematician chooses two values, i.e. two false positions: x_1 and x_2 here. She calculates the answer they would give and notes down the difference between the two answers (here e_1 et e_2), [2] compared to the value she is looking for. She adds or subtracts the cross-multiplied products, depending on whether the errors have the same sign. She then calculates the sum total or the difference between the two errors, again depending on whether they have the same sign or not. The division by this value gives the correct answer. In algebraic notation – which is clearly not well suited to describing such a method! – the following formula is applied:

$$x = \frac{|e_2 x_1| \pm |e_1 x_2|}{|e_2| \pm |e_1|} \qquad (1)$$

The first thing one notices in the text is the disproportionate size of the + and – signs. Moreover, this is the first time they are used, after a hundred pages of arithmetic from which they are entirely absent.

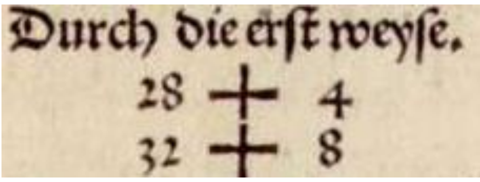

***Figure 1:* Disproportionate size of the + sign,**
which, furthermore, does not function as an operator.

When one deciphers the text, it becomes apparent that the signs are used as indicators, as a kind of stenography to show that there is an excess or a deficit. They do not, therefore, mean anything in an operational sense.

After his explanation, Schreiber moves on to an example ... but does not feel the need to elaborate on it. That bears witness to how well known false position was among his contemporaries: he doesn't go into details about what everyone already knows. So, five centuries on, it's our job to explain what he overlooks!

Schreiber suggests trying to find a number which, when divided by 4, multiplied by 8 and halved, gives 24. For those of us trained in algebra, the answer is obvious: 24 [(x : 4 × 8) : 2 = 24].

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2. [Translator's note] This difference – whether an "excess" or a "deficit" – is known as an "error" (*e*). See Randy K. Schwartz, "Adopting the Medieval 'Rule of Double False Position' to the Modern Classroom", in *Mathematical Time Capsules: Historical Modules for the Mathematics Classroom*, Mathematical Association of America, 2010.

We can also see that this problem goes round in circles: no matter what the value chosen for the false position, the answer will always be the same value. But this wasn't true of 16th-century readers. To whom the author suggests first trying with 28.

Let's suppose, then, that the number sought is 28. We take one quarter – 7 – and multiply it by eight, giving 56, which we divide by two, giving 28. As the answer is 28, there is an excess of 4 vis-à-vis the 24 we were trying to obtain. Schreiber condenses this to "28 + 4", which, more long-windedly, means that "If one assumes *twenty-eight*, there is an *excess* of *four*". The second attempt assumes 32, which gives 32 – as *we* would expect – and an excess of 8, which the medieval reader is supposed to work out. Hence the precis "32 + 8", meaning "If one assumes *thirty-two*, there is an *excess* of *eight*".

As there are two excesses, subtractions are required in formula *(1)*: (28 x 8) *minus* (32 x 4) in the numerator, and 8 *minus* 4 in the denominator, giving 96 / 4 and an answer of 24.

Double false position

In the slightly more complex double false position, the mathematician assumes two successive false positions and calculates the difference (error) between each incorrect answer and the answer he is seeking. Then he applies *either* rules of proportionality to the errors *or* a formula (which can be proved as its equivalent[3]), which is what Schreiber does here. And that produces the correct answer.

Choosing which version depends on the problem, as we shall see. But we also show that double false position is applicable in all cases, which means that some ancient authors, like Schreiber here, consider this method alone.

In the words of Leonardo of Pisa (also known as Fibonacci) in 1202, the formula takes the following form: "The first error is multiplied by the second position; and the second error by the first position. And if the errors are both diminution, or both augmentation,[4] the smallest sum of the said multiplications is subtracted from the larger; and the remainder is divided by the difference between the errors. Thus a solution to the questions is found: and if one error is diminution, and the other augmentation, the two multiplications are then added together, and the sum is divided by the sum of the errors."[5]

3. Gavin & Schärlig, *Longtemps avant l'algèbre, la fausse position*, pp. 22–23.
4. In other words, if there are two *deficits*, or two *excesses*.
5. Leonardo of Pisa (Leonardo Pisano), *Liber Abaci*, Pisa, 1202, folio 141^r of the manuscript, and page 319 of the transcription by Baldassare Boncompagni: *Scritti di Leonardo Pisano*, volume I, *Il Liber Abbaci di Leonardo Pisano pubblicato secondo la lezione del codice Magliabechiano C. I, 2616, Badia Fiorentina, N° 73*, Rome, 1857, 459 p.

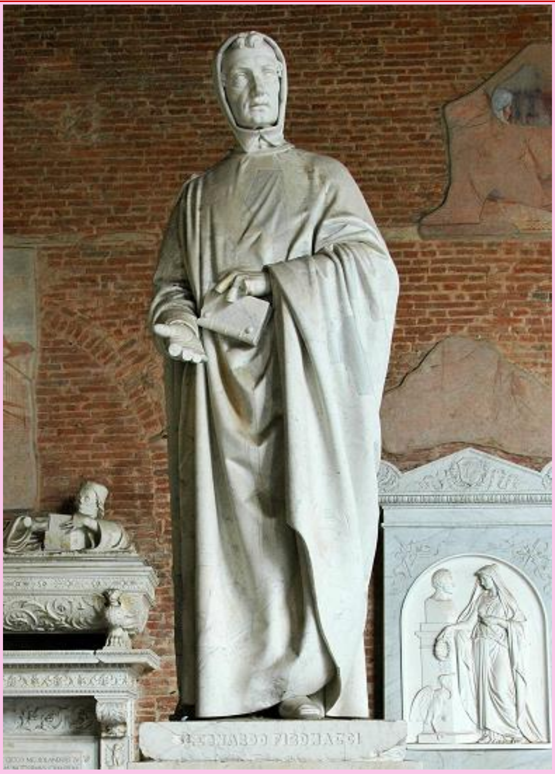

Figure 2: ***Statue (1863) of Leonardo de Pisa, or Fibonacci, at the Camposanto de Pise (Italy)***
(photo WikiCommons Hans-Peter Postel, cc-by 2.5)

Here is an example – the same one as in the panel above, but with a slight alteration that makes it necessary to use *double* false position. (The ancients – who did not know about algebra – failed to see that all they needed to do was transfer the constant 5 to the other side of the = sign.)

A man has found some treasure. He has squandered half of it, replaced a quarter of it, then squandered an eighth of it *and, lastly, recovered 5 gold coins*. He has 65 gold coins left. What was the treasure to begin with?
Let's start off with simple false position. First with a false position of 8: 8 – 4 + 2 – 1 + 5 = 10. Then with a false position of 16: 16 – 8 + 4 – 2 + 5 = 15. As we can see, the false position has been doubled, but the answer has not! The problem is therefore no longer linear.

One therefore has to use double false position and the formula:
First false position: 8. The answer is 10, so 55 are coins missing.
Second false position: 16. The answer is 15, so 50 coins are missing

$$treasure = \frac{(16\times55)-(8\times50)}{55-50} = 96\ golden\ coins$$

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Here is another example (China, 2nd century BCE). Let's suppose there is a low wall 90 cm high, on top of which a melon plant is growing at a rate of 7 cm a day, and at the bottom of which there is a calabash vine growing at a rate of 10 cm a day. How many days will it take for them to meet?

First false position: 5 days. The melon plant has grown downwards by 35 cm, and the calabash vine upwards by 50 cm. The total is 85 cm, so there is a deficit of 5 cm. Second false position: 6 days. The melon plant has grown downwards by 42 cm, and the calabash vine upwards by 60 cm. This gives a total of 102 cm: there is an excess of 12 cm.

By applying the formula given in words by Leonardo, but now written in modern-day notation, we calculate the number of days:

$$Number of\ days = \frac{(6 \cdot 5) + (5 \cdot 12)}{5 + 12} = \frac{90}{17} = 5\frac{5}{17}$$

As we can see, the formula is more elegantly expressed in writing (which is what Leonardo does) than translated into modern-day notation. The same is true of formula *(1)* above.

As his example has produced two excesses, and therefore subtractions when the formula is applied, Schreiber repeats the same problem with different false positions in order to produce an excess and a deficit, and thereby illustrate how the mechanism works with additions. This is what he calls a second way. He chooses 21 then 28, which produces a deficit of 3 and then an excess of 4, as the reader of the copy digitised for this site rightly corrected. The author himself makes a mistake, and finds a deficit of 12 and an excess of 8, which he summarises as "21 – 12" and "28 + 8". And it certainly was the author's mistake, for the chances of the typesetter making two – and what's more proportional – misprints in such quick succession are very slight.

The numerator of the formula is therefore (28 x 3) *plus* (21 x 4), and the denominator is 3 *plus* 4. This gives 168 / 7, and again an answer of 24.

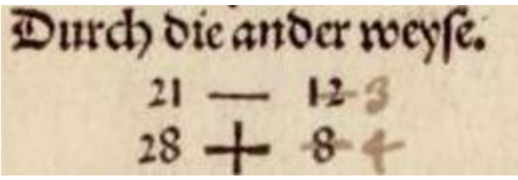

Figure 3: With 28, there is an excess of 4 (after correction), as in Figure 1; with 21, there is a deficit of 3.

The "double false position" calculation reconstructed

For those who want a "demonstration" of formula *(1)* in modern terms, let's take the problem $Kx + b = c$, with A_1 and A_2 as the two false positions:

$$KA_2 + b = c + \delta_2$$

We multiply the first equality by δ_2 and the second by δ_1:

$$KA_1\delta_2 = (c - b)\delta_2 + \delta_1\delta_2$$
$$KA_2\delta_1 = (c - b)\delta_1 + \delta_1\delta_2$$

We subtract the second equality from the first:

$$K(A_1\delta_2 - A_2\delta_1)= (c - b)(\delta_2 - \delta_1); \text{ hence } K\frac{A_1\delta_2 - A_2\delta_1}{\delta_2 - \delta_1} + b = c$$

So, the solution to $Kx + b = c$ is given by $\frac{A_1\delta_2 - A_2\delta_1}{\delta_2 - \delta_1}$, which corresponds to formula *(1)*.

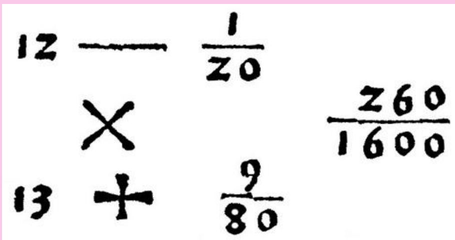

Figure 4:* *Widmann's* Algorismus Ratisbonensis *(c.1450), folio 137r.
In the problem he sets out, position 12 gives a deficit of 1/20, while position 13 gives an excess of 9/80. The difference between the two errors is 260/1600, and the cross is a reminder to "cross the product" in the denominator.

The reader might like to try using false position to solve the so-called Diophantine problem, which asks how long the following person has lived:

His childhood represented a sixth of his life; a twelfth was given over to adolescence; he married after a seventh of his life; five years later he had a son, who died having reached half the age of his father, who himself died eight years after this death.[6]

Taking the two false positions $A = 72$ and $B = 120$, for example, one finds:[7]

$$72 \rightarrow 74^{4/7} \text{ so } \delta_A = 2^{4/7}$$
$$120 \rightarrow 112^{2/7} \text{ so } \delta_B = -7^{5/7}$$
$$x = (72 * 7^{5/7} + 120 * 2^{4/7})/(7^{5/7} + 2^{4/7}) = 864 / (10^{2/7}) = 84$$

6. A problem included in Gavin & Schärlig, *op. cit.*, p. 7. The reader will quickly notice that it is a $Kx + b = c$ type problem.
7. The fractions below are not exponents but a shorthand to show, for example, 74 + 4/7 (for which decimal expression serves no purpose).

INTERLUDE: THE + AND – SIGNS

As we have already noted, Schreiber does not use the + and − signs in arithmetic, but only to mark false positions. And, as we will see, it is in algebra alone that he gives them an operational role.

Let's briefly recall the history of these two symbols before Schreiber, starting with manuscripts. The + symbol appears in the *Algorismus Ratisbonensis*, written around 1450. The + and – symbols are included in the *Codex Dresden C 80*, written in 1481, but only as corrections, which some attribute to Johannes Widmann (*c*.1460–1498). Finally, the *Codex Lips. 1470*, conserved in Leipzig and containing a series of lectures delivered by Widmann in 1486, contains the two symbols in the text. In a word, it is in these three manuscripts, considered as a whole, that the two symbols appear for the first time, and Widmann played a part in that.

Moving from manuscripts to books, Widmann's publication is the first that we should consider. It was published in Leipzig in 1489 under the title *Behende und hübsche Rechenung auff allen kauffmanschaft* ("Fine and Nimble Calculation for All Merchants"). The opening of the book is given over to arithmetic, as one would expect, but we have to wait until the 88th folio to see a – and + sign, both of which are operational. There are only a dozen more in the rest of the book. Widmann was therefore the first to use these symbols, with this meaning, in a book.

And that's not his only masterstroke. On folios 201 and 202 he discusses false position. And in the diagrams he provides to summarise a problem, he again uses + and –, but this time as *markers*, to indicate an excess or deficit.

Schreiber therefore takes up Widmann's invention, whose for the notation allows him to symbolically illustrate the practice of false position. His own revolution would be to use the same in symbols in an algebraic context.

***Figure 5:* Widmann's work.** *Left: title page of the original edition, Leipzig, 1489 (NB: the first works printed by Gutenberg date from 1452 to 1455); right: illustrated title page of the Augsburg edition, 1526 (images MDZ, Bayerische StaatsBibliothek, see here).*

ADDITION IN ALGEBRA

This extract also reveals another innovation: algebra as it was presented to German readers in 1521. It's a very fine little example of literal calculus. Though its notations seem almost unbearably unwieldy to us, it contains a revolution, a world first: operational + and – signs, very clearly described from the outset as *und* ("and") and *mynnder* ("less"). The use of these signs was an important step towards algebra becoming an effective mathematical technique. The irony of fate being that this would slowly kill off false position, which was at the origin of the systemic use of the two signs.

Let's first deal with those unwieldy notations. They show just how long it took for algebra to gain its status as the mathematical instrument *par excellence*: at the time, it was not an ideal method to solve a problem – far from it!

Schreiber indicates a constant by placing a letter N and a colon after it, e.g. "6N:", where we would simply write "6". He indicates a first-degree variable, which for us would simply be "x", by writing *pri* (for *prima*) and a colon after it, e.g. "9pri:", where we would write "9x". Later on – in other passages – he indicates a second-degree variable by *se* (for *secunda*), a third-degree variable by *ter* (for *tertia*), and so on and so forth.

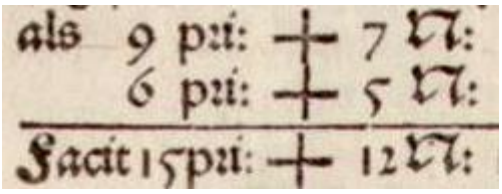

***Figure 6:** The first of the two additions.*

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The lesson begins by adding two binomials with a + sign in the left column, and two other binomials with a – sign in the right column. Written our way, these two additions would be presented like this:

$$(9x + 7) + (6x + 5) = 15x + 12$$

$$(6x - 4) + (8x - 10) = 14x - 14$$

The author then adds together two other binomials with a + and – sign, one of which produces a positive constant, and the other a negative constant:

$$(6x + 6) + (12x - 4) = 18x + 2$$

$$(4x + 2) + (6x - 6) = 10x - 4$$

He ends by adding together two new binomials, which again have opposite signs but where the – sign is in the first and the + sign in the second:

$$(6x - 4) + (6x + 2) = 12x - 2$$

$$(6x - 2) + (6x + 4) = 12x + 2$$

Rather than elaborating a general rule – which he presents at the end of the passage – Schreiber deals with each case by giving an example, thereby demonstrating real pedagogical flair.

SECOND INTERLUDE: HOW OLD IS ALGEBRA?

This is a good point to briefly consider an intriguing question in the history of mathematics, namely how long algebra has existed as an *effective* instrument in *our* area of the world. Whenever one dates this maturity back to the end of the 16th or beginning of the 17th century – for example, in a lecture alluding to the progressive demise of false position – audience members invariably make an objection: "Impossible! Algebra has been around since the writings of Al-Khwarizmi, which date from 825!"

But this ignores the fact that the great man's text was written in Arabic and "published" around 825 in *Baghdad*, at a time – the 9th century – when European countries were in the midst of what some are loth to call the Dark Ages, but which, from the point of view of mathematics in any case, were not enlightened.

The first documented appearance of algebra in Europe – in Italy, in fact – can be found in Leonardo of Pisa's *Liber Abaci* ("Book of Calculation"), dating from 1202, i.e. the 13th century. In Germany, Schreiber had presented algebra in 1521, as we have just seen, and in this he was followed by Rudolff in 1525, and then by Stiefel, who, in 1591, introduced some order into Stiefel's work. This already brings us up to the 16th century. And the same goes for France, with Scheybel in 1550 and Viète in 1591 – again, at the end of the 16th century.

But one still needs to consider the *form* these appearances took. With the likes of Schreiber's *pri:* and *se:* and the textual developments of other authors – where successive developments superseded one another in new and innumerable notations – algebra, it has to be said, was still in need of fine-tuning. Only at the end of the 16th century and the turn of the 17th century – with the advent of *x*, bare constants and, what's more, the invention of the equals sign by the Welshman Robert Recorde – did algebra finally become, in *our* area of the world, the *fully effective* instrument we know now it to be.

MULTIPLICATION IN ALGEBRA

Another extract complements the previous extract on addition, showing how Schreiber sets about presenting multiplication in algebra. Here again, he reviews all the possibilities: he calculates the products of two binomials containing a + sign, then the products of two binomials containing a – sign, and finally those of two others containing opposing signs. Each time, he announces the rule that applies to the signs. Transcribed in modern-day notation, his three multiplications would be written as follows:

$$(6x + 6)(5x + 8) = 30x^2 + 30x + 48x + 48 = 30x^2 + 78x + 48$$

$$(6x - 8)(5x - 6) = 30x^2 - 40x - 36x + 48 = 30x^2 - 76x + 48$$

$$(6x + 8)(5x - 7) = 30x^2 + 40x - 42x - 56 = 30x^2 - 2x - 56$$

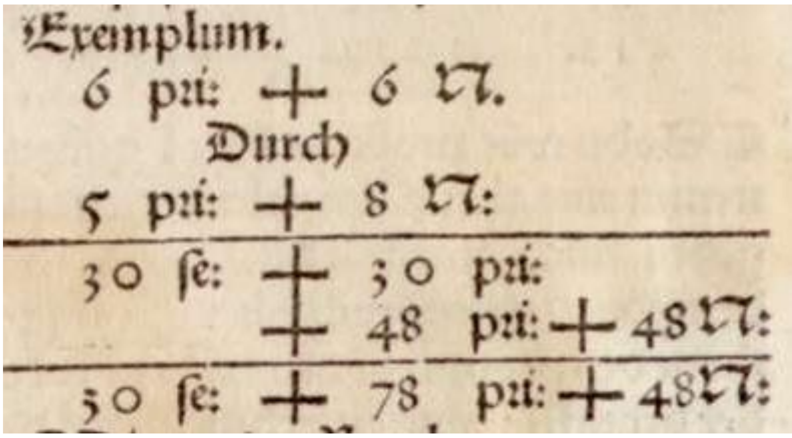

***Figure 7:* The first of the three multiplications.**

He concludes this passage with a "rule of signs" … that is a little more long-winded than the one taught in our schools today:

> *When the signs are the inverse of one another, subtract the smaller from the bigger, and to what remains write the sign of the biggest number.*

AN EXAMPLE CALCULATED WITH FALSE POSITION AND WITH ALGEBRA

Along with several others given by the author, this example illustrates how to use the two (then rival) methods to solve a problem: false position on one hand, and the "first rules of algebra" on the other. And, once again, false position gets to go first.

By choosing two false positions, 300 followed by 240, Schreiber obtains two excesses, 77 and 57 respectively. He summarises the situation by using two + signs as markers:

$$300 + 77$$
$$240 + 57$$

Though it seems obvious to us, this is not a case of two elementary additions, but a stenographic precis of one stage in the solution to the problem. The author then expedites the process by directly giving the formula's numerator, divisor, and answer. He considers this development – which we would write as

$$\frac{(240\cdot 77)-(300\cdot 57)}{77-57}=\frac{18480-17100}{20}=\frac{1380}{20}=69$$

– to be obvious, and doesn't bother explaining why you need to reverse the order of the two products of the numerator, by virtue of the rule "Subtract the smaller from the bigger".

The algebraic solution, with its never-ending unwieldy notations, gives some insight into the impracticality of this method in 1521. The notations can be compared with what a newcomer to algebra, scrupulously noting down all the stages, would come up with today:

$$\frac{\left\{\left[\left(\frac{2}{3}x\right)\cdot 4\right]+8\right\}\cdot\frac{1}{2}}{4}-4=20$$

$$\frac{\left\{\left[\frac{8}{3}x\right]+8\right\}\cdot\frac{1}{2}}{4}-4=20$$

$$\frac{\left\{\frac{8}{3}x+8\right\}\cdot\frac{1}{2}}{4}-4=20$$

$$\frac{\frac{4}{3}x+4}{4}-4=20$$

$$\frac{1}{3}x+1-4=20$$

$$\frac{1}{3}x=23$$

$$x=69$$

This illustrates perfectly what we noted above: in 1521, algebra still had some progress to make before it could be considered an efficient mathematical instrument!

NOTHING BEATS A VOYAGE THROUGH TIME AND SPACE!

This article has introduced the reader to an undeservedly unsung text. It illustrates an important rule of ours: in the history of mathematics, nothing beats a voyage through time and space. One works one's way back to the source, and gets a taste of those very peculiar ancient notations (in this case, those of Early Modern High German) to boot.

But the above discussion contains another, perhaps more important, lesson, one that shows us why algebra took so long to establish itself on our shores. Not only was it unwieldy to use in its contemporary form. Above all, it faced competition from false position, which proved a far more elegant solution to linear problems.

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(November 2014)

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