Leonardo da Pisa: A minor yet famous maths problem and a significant yet forgotten contribution to the arrival of algebra in Europe

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Leonardo da Pisa (*c*.1175–*c*.1250) was undoubtedly one of history's great popularisers of mathematics, in the noble sense of the word: someone who makes a discipline accessible to the greatest number of people and thereby contributes to its dissemination throughout society.

Yet his fame among keen mathematicians today – who know him by the posthumous nickname Fibonacci – rests on an unprepossessing sequence of numbers, hidden in one of his countless entertaining maths problems, and whose remarkable properties he himself did not even notice.

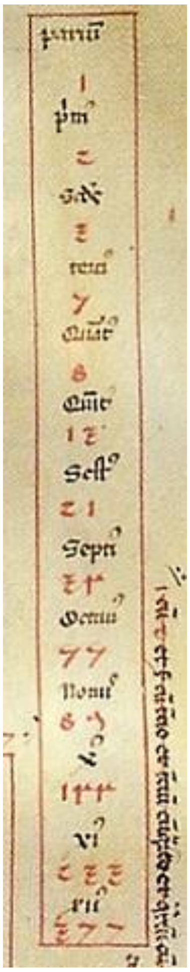
This is unfair. Leonardo da Pisa deserves universal recognition for his huge contribution to the development of calculation in the Western world. But the oversight can also be explained, at least in part. For Leonardo's *Liber abaci* – an opus written in Latin and concluded in 1202 – is currently translated into just one language: English. And even this translation was not published until 2002. It is therefore worth offering French-speaking readers an overview of this work. To do so we have translated and commented on five noteworthy extracts from the text.

[Extract 1] **He wasn't called Fibonacci!**

The title tells us that this book is a *liber abaci*, i.e. a book of calculation or arithmetic. It is worth noting the use of the word *abacus* here: this is a reference to the ancient method of performing calculations using a counting frame strung with beads arranged into columns. The new and much more efficient numerals and method of calculation described by Leonardo would slowly replace this technique, but its name lived on. It should also be noted that the title of the book is usually spelled *Liber abaci*, which is preferable to the alternative and almost equally common spelling *abbaci*.

The title then tells us that the book was "composed by Leonardo, son of Bonaccio, a Pisan" (*compositus a leonardo filio Bonacij Pisano*). The name *Fibonacci* is nowhere to be seen. Yet it is on folio 123 of this manuscript that the rabbit problem – which would give rise to the famous "Fibonacci sequence" –makes its first appearance. As we will see, this term was not once used during Leonardo's lifetime.

Figure 1**: Extract from the* **Liber Abaci, held at the manuscript department of the Biblioteca Nazionale Centrale di Firenze (see here; 214 bound pages, 29.8 x 20.8 cm).* *The right-hand page shows the "Fibonacci sequence", a section of which is shown enlarged below, in* ***Figure 1a****. The sequence is 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610 (each number is the sum of the two previous numbers). The figures are not written as we are used to seeing them today: the 3 looks like a Z, the 5 looks like a 7, and the 4 is difficult to describe …*



Fibonacci wasn't even Leonardo's nickname. He was sometimes referred to as *bigollo*, which might have meant "traveller[1]" or "fidget" in Pisan dialect, or alternatively "good for nothing" or "blockhead". Some commentators think that the nickname may have been a self-mocking invention of Leonardo himself, while others see it as an insult that he could not shake off:[2] his fellow countrymen, it is argued, may have taken to calling him by this name because, unlike them, he never fully devoted himself to the business of trade. He is memorably described as *Leonardo bigollo quondam Guilielmi* ("Leonardo Bigollo, son of Guglielmo) in a document dating from 1226,[3] and he is mentioned as *Maiestro Leonardo Bigollo* ("Master Leonardo Bigollo") in the Pisan authorities' decision, in 1240 or 1241, to pay him a pension for the services he had provided. This made the nickname "official".

1. Smith and Karpinski (1911), *The Hindu-Arabic Numerals*, p. 130.
2. Such as Boncompagni (1852), *Della vita e delle opere di Leonerdo Pisano…*, pp. 16 ss.
3. Smith and Karpinski (1911), *The Hindu-Arabic Numerals*, p. 129.

Another clue suggesting that Leonardo was not called Fibonacci, not even as a nickname, can be found a half-century later on: an anonymous author, known as *Maestro umbro* (The Umbrian Master) – so called because he wrote his text in the dialect of Umbria – prefaced his own work of arithmetic, written around 1290, with the title "This is a book of calculation according to the opinion of Master Leonardo of the house of the children Bonacci of Pisa" (*Quisto ène lo livero de l'abbecho secondo la oppenione de maiestro Leonardo de la chasa degli figluogle Bonaçie da Pisa*). If Leonardo had been (nick)named Fibonacci, it would no doubt have been mentioned in this title, in lieu of this allusion to the house of Bonacci.

It was probably the phrase *filio Bonacij*, which appears in the title of the *Liber abaci*, that led some authors, apparently from the late 17th century onwards, to attribute Leonardo the homophonic surname Fibonacci. Boncompagni[4] finds mentions of the name in 1787, then in 1812, 1818 and 1820, and later in the writings of the mathematician Michel Chasles in 1837. In the same year, Guillaume Libri[5] also cites Fibonacci and goes as far as to provide an explanatory note: according to Libri, this name is a contraction of *filius Bonacci* – "a contraction which is common in the formation of Tuscan family names". Thus although Libri was not the first, he seems to have been aware that the name was not directly linked to Leonardo, and that he would have to justify his reasons for using it.

The title of the manuscript, in the edition transcribed and printed by Boncompagni in 1857, states that the work was written "in the year 1202" (*in Anno .M^{o} cc^{o} ij^{o}.*). Other manuscripts bear this same title, supplemented with the phrase "and corrected by himself [Leonardo] in [12]28" (*et correctus ab eodem anno XXVIII*).[6] The dedication in the first line to Michael Scot, astrologist to the Emperor Frederick II – a dedication which, some argue, was not added until the second edition – suggests that the text we use today was revised by Leonardo in 1228, and that it was therefore the definitive version. According to Boncompagni, the copy he himself used was produced in the 14th century.[7]

4. Boncompagni (1852), *Della vita e delle opere di Leonerdo Pisano…*, pp. 8–10.
5. Libri (1837), *Histoire des sciences mathématiques en Italie*, pp. 20–21.
6. Boncompagni (1852), *Della vita e delle opere di Leonerdo Pisano…* , pp. 73–74.
7. *Ibid*., p. 32.

***<u>Figure 2</u>*: Statue of Leonardo da Pisa (known as Fibonacci) by Giovanni Paganucci (1863), at the Campo Santo** *(WikiCommons; author: Hans-Peter Postel)*

[EXTRACT 1, CONT.] **HE GOT AROUND!**

Leonardo grew up in Pisa and in Bugia in a world of trade. It is hardly surprising, then, that when his father saw the Arab tradesmen doing their calculations, he realised that this technique represented a considerable commercial advantage and that it would useful in his son's future career if he learned to do the same. But Leonardo went a great deal further than that: he set off to travel the Arab world – and he did not keep what he learned during his travels to himself. Instead he first developed, and then published, his discoveries – to the great benefit of Tuscan traders. The latter soon abandoned the old method of calculating with beads on an abacus and chose to send their children to *bottege d'abbaco* (shops where arithmetic was taught). There they could learn the new method of calculation. And what made it so original was that it could written down.

Leonardo's studious voyage abroad was unusual for its time. And it gave rise to a most significant text in which Leonardo presents the development of arithmetic over four centuries, since al-Khwārizmī had laid down its foundations on a piece of parchment in Baghdad in 825.[8] In comparison, the translation of the latter by Toledo monks in around 1143, or a more abridged but better written version produced by Sacrobosco in Paris, circa 1230, covered no more than the Arabic numerals and the four arithmetic operations – i.e. the state of arithmetic as it had been in the days of al-Khwārizmī, four centuries earlier. The difference was therefore substantial – and Leonardo was well aware of the importance of the work of the mathematician of Baghdad.

Leonardo was one for "frank discussions" and explicitly spoke out against the use of abaci to perform calculations, which he called the "Pythagorean arc" in reference to a text by Gerbert d'Aurillac[9] produced two centuries previously. He also rejected *algorism*, which is quite simply the very rudimentary method of calculation described in Sacrobosco's text, whose name is derived from a Latin deformation of al-Khwārizmī's name (*algorismus*).

Lastly, he declares his support for the *Indian approach*, which, as we will see in a moment, consists in using what we call Arabic numerals – incorrectly, in fact, because they are Indian in origin – and more generally in *writing down* one's calculations.

These seemingly inconsequential words actually amount to a profession of faith from the man who would introduce the new method of calculation to the West. But they do not prevent him expressing his admiration for Euclidean geometry, to which he had doubtless been introduced by the Arab scholars he met during his travels. Nor do they prevent him using *Roman* numerals to announce that he will divide his book into XV chapters! Indeed, Roman numerals can be found scattered throughout his text.

8. See Schärlig (2010), *Du zéro à la virgule*, pp. 42–58.
9. See Schärlig (2012), *Un portrait de Gerbert d'Aurillac*, pp. 89–90.

How do you spell *al-Khowarizmi*?[10]

This is a good moment to make a point about spelling. We have opted to use the spelling *al-Khowarizmi*, first because it is the most common spelling of this name in French, and secondly because it is recognisably related to the words *algorism* and *algorithm*, which are derived from this name and which, when they entered the Latin and then French languages, where spelled with an *o*.

It is true that, in view of the conventions that have been slowly established to transcribe the place names of this region, it would be more logical to write *al-Khwarizmi* – following Allard's example[11] – because we now use the spelling *Khwarezm* for the town or region he came from (though one does occasionally see the spelling *Khorezm*, in which the *o* reappears!). The Germans have fewer problems: they write *Al Chwarizmi* for the man and *Chwarizm*[12]for the place he came from (though Yushkevich's translator chose *Choresm*,[13] which takes us back to where we started!).

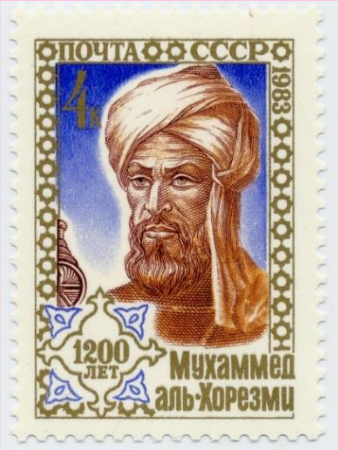


***Figure 3*: A four-kopek stamp issued in the USSR in 1983 to mark what is thought to be the 1200th anniversary of the birth of al-Khwārizmī.** *Note that the Russians spell the name with an* о*, as we do in French, unlike the German and English languages, which do away with this letter. On the other hand, the* wa *is notably absent. This Arabic sound is transcribed rather haphazardly in all Western languages.*

[EXTRACT 1, CONT.]

He introduced the West to the new method of calculation

This is Leonardo da Pisa's main claim to fame: it was he who did most to introduce the West to Arabic numerals. But the numerals themselves were only

10. This panel refers to the French spelling of the mathematician's name. In English he is known as al-Khwārizmī.--Trans.
11. Allard (1992), in the title of his book.
12. Vogel (1963), p. 42.
13. Juschkewitsch (1964), p. 186.

the start: their great strength was that they made positional notation[14] possible, which itself allowed for written calculation, which was the basis for the development of our scientific culture. It was this method of written calculation – which we have named the *new method of calculation* – that the Pisan handed down to us in a manuscript 214 pages long.

It is important to understand that Arabic numerals and the new method of calculation very much went hand in hand. In themselves the numerals were not a major step forward, even though they induce admiration among our contemporaries: in Leonardo's day, Roman numerals were still a sound way of noting down numbers (indeed, some book-keepers continued to use Roman numerals until the 17th century). Positional notation simply had the advantage of being a little more concise. For most tradesmen at least, written calculation in itself was no more useful than using an abacus. It was the *combination* of these two discrete actions – noting down numbers and performing a calculation – which produced such a tour de force. Written calculation *required* positional notation, and positional notation allowed calculation to be taken much further than the four arithmetic operations.

Thus, though Leonardo did not invent anything, he was vitally important because he bequeathed us nothing less than our entire system of arithmetic.

For the record, it is worth noting that Leonardo writes his numbers – *Indian figures*, as he calls them – right to left. In other words, he follows the reading conventions of the Arabic language rather than of Latin. Furthermore, he uses only nine numbers: for him, the Arab *zephir* is not a number but a sign showing that in a given position there is nothing. As we will see, his description of positional notation is also rather clumsy.

It is also worth mentioning that at the time of Leonardo's travels, the Arabs were quite open about the Indian origins of their numeral system. This is still the case … except where religious extremism is involved.

14. Our number system is described as positional, because, in a number such as 423, for example, we know that there are two tens because 2 is in the "tens" position.

***Figure 4*: A reproduction by the University of Mainz (1977) of a bronze Roman counting frame (abacus) held at the Bibliothèque Nationale de France in Paris**
(WikiCommons, photo: Mike Cowlishaw)

[EXTRACT 2] *HIS* SEQUENCE: ONE PROBLEM AMONG A HOST OF OTHERS

Leonardo da Pisa's name is largely unknown. Yet as anyone keen on mathematics knows, the name Fibonacci – which was attributed posthumously, as we have seen – immediately brings to mind something known as the *Fibonacci sequence*, the origins of which are worth considering here.

The Fibonacci sequence is basically an entertaining problem about breeding rabbits. It takes up fewer than two pages of Leonardo's manuscript, but, seven centuries later on, it would inspire another author – who spotted the famous sequence. Yet our Pisan does not refer to at all!

The problem is lost in a mass of other problems. To give readers a sense of this, we have chosen to cite the two problems that precede it as well as the one that comes immediately afterwards. The first one is a story about bread and underlines the great subtlety of Leonardo's mathematics. The second is about perfect numbers, then comes the problem about rabbits, followed – this time without a subtitle – by a problem that involves finding out how much money four men have in their purse (knowing how much they have when they are considered in threes). This is what you might call cheerful disorder, and the aim was clearly to cheer up the book's readers!

[EXTRACT 2, CONT.] **OBVIOUS, BUT WRONG!**

Though they aren't our main concern, let's take a brief look at the first two problems, because they show just how much pleasure Leonardo took in explaining his arithmetic.

On Two Men Having Bread

There were two men, the first of whom had 3 loaves of bread and the other 2 loaves, and they took a walk to a certain fountain where they met together sitting and eating, and a soldier passed by; they invited him to join them, and he sat down and ate with them, and when they had eaten all the bread the soldier departed leaving them 5 bezants for his share. Of this the first took 3 bezants as he had three loaves; the other truly took the other 2 bezants for his two loaves. It is sought whether the division was just or not. A certain person asserted that the division was correct as each had one bezant, for each loaf, but this is false because the three ate all five loaves. Whence each took 1 2/3 loaves; the soldier ate 1 1/3 loaves, that is 4/3, from the loaves which the three had. Of the loaves truly the other ate only so much as 1/3 of one loaf. Therefore the first man took 4 bezants and the other 1 bezant.

What is remarkable about this problem is that, as a modern reader, one is just dying to say that the suggested solution is correct, because intuitively it seems fair. But it isn't that simple. In modern mathematical language, the solution is that every man eats 5/3 of the bread. So, the first man has given 3 loaves of bread minus 5/3 of the loaves, i.e. 4/3 of the loaves. The second has given 2 loaves minus 5/3 of the loaves, i.e. 1/3 of the loaves. The first man has therefore given four times as much as the second man. The correct division is therefore 4 bezants for the first man and only one for the second man.

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The second problem presents a way of finding perfect numbers:

On the Finding of Perfect Numbers

A number is perfect when the sum of its integral factors is the same number; as 6, which has [the] *factors 1/6, 1/3, 1/2, and no other integral factors. And taking 1/2 and 1/3 and 1/6 of 6, namely 3 and 2 and 1, undoubtedly their sum is 6, and the 6 is found thus: you double 1; there will be 2, and you double the 2; there will be 4, from which you take 1; there remains 3 which is a prime number, that is it has only the factor 1; you multiply it by half of the abovewritten 4, and thus you will have 6. Whence if you will wish to find another perfect number, then you will double again 4; there will be*

8 from which you take 1; there will remain 7, which is a prime number; you will multiply it by half of the 8, namely by 4; there will be 28, which is a perfect number because it is equal to the sum of its factors. The factors are indeed 1/28, 1/14, 1/7, 1/4, 1/2. Again doubled 8 makes 16, from which is subtracted 1; there remains 15, which is not a prime number; you will double again 16; there will be 32 from which you take 1; there will remain 31, which is a prime number; you will multiply it by the 16, and you will have another perfect number, namely 496, and always doing thus you will be able to find perfect numbers without end.

The method used here is one described by Euclid,[15] which shows Leonardo's fondness for this author. In modern mathematical language, if $2^n - 1$ is a prime number, then $2^{n-1} \times (2^n - 1)$ is a perfect number. For example: $2^2 - 1 = 3$ is a prime number, so $3 \times 2 = 6$ is a perfect number. Or if $2^3 - 1 = 7$ is a prime number, then $7 \times 4 = 28$ is a perfect number. But $2^4 - 1 = 15$ is not a prime number, so the rule does not apply. On the other hand, $2^5 - 1 = 31$ is a prime number, so $31 \times 16 = 496$ is a perfect number. Leonardo stops there, but not before pointing out that the method produces perfect numbers to infinity.

[Extract 2, cont.] **Leonardo: Much better than Fibonacci!**

But let's get back to our rabbits.

How Many Pairs of Rabbits Are Created by One Pair in One Year

A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also. Because the abovewritten pair in the first month bore, you will double it; there will be two pairs in one month. One of these, namely the first, bears in the second month, and thus there are in the second month 3 pairs; of these in one month two are pregnant, and in the third month 2 pairs of rabbits are born, giving 5 pairs that month, which that very month bear 3 pairs. In the fourth month there are 8 pairs, of which 5 pairs bear 5 more pairs; added to the 8 pairs, this gives 13 pairs in the fifth month, of which 5, born this very month, do not give birth this same month, but the other 8 couples bear; and thus the sixth month we have 21 pairs, which added to the 13 pairs born in the seventh month, give 34 couples this month, which added to the 21 pairs born in the eighth month give 55 couples, which added to the 34 couples born in the ninth month, give 89 pairs that month, which added to the 55 pairs born in

15. Euclid, *Elements*, Book IX, proposition 36.

the tenth month, give 144 pairs that month, which added to the 89 pairs born in the eleventh month, give 233 pairs that month. Which added to the 144 pairs born the previous month, give 377 pairs, and that is how many pairs the original pair bore in this enclosure in one year. You can see in the margin how we have arrived at this calculation, that is to say how we have added the first number to the second, that is to say 1 to 2; and the second to the third; and the third to the fourth; and the fourth to the fifth; and so on and so forth, until we add the tenth to the eleventh, that is to say 144 to 233, and we have obtained the sum of rabbits written above, that is to say 377. You can proceed in this way with an infinite number of months.

This is a typical entertaining maths problem. We gradually arrive at a sequence of numbers and are astonished to find that there are 377 pairs by the end of the year. The mission is accomplished, and Leonardo stops there and moves on to the next problem. He doesn't point out that the numbers we have gradually obtained form a *sequence*, which was perhaps obvious to him but irrelevant to the matter at hand.

It was the mathematician Édouard Lucas (1842–1891), a specialist of number theory, who brought this calculation – now known as the *Fibonacci sequence* – to mathematical fame at the end of the 19th century. When the number corresponding to stage *n* is noted F_n (F signifies *Fibonacci number*), the subsequent numbers in the sequence will be defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$.

***Figure 5*: Édouard Lucas (1842–1891)**

What made Leonardo da Pisa famous, then, was a sequence he didn't spot and a name that wasn't really his! Yet he left a much more significant legacy behind him than this sequence, namely the whole of his *Liber abaci*. Curiously, though, this magnum opus did not bring him as much fame as his little rabbit problem.

[EXTRACT 3] **HE INTRODUCED ALGEBRA INTO THE WEST**

One aspect of this legacy, we have already mentioned, is the introduction of the new method of calculation to the West. But that wasn't all. Leonardo also made another contribution. This is the third part of Chapter 15 of the *Liber abaci.* This chapter is devoted to the "Pertinent Geometric Rules and […] Problems of Algebra and Almuchabala", and the third part of it covers algebra – for the first time in the West.

It is true that on this particular point, the author's pre-eminence is rather symbolic, because, as we will see, his algebra is not at all operational. But this doesn't detract from the central fact: he dared something new. And he was certainly conscious that he was ratcheting up the difficulty of his subject matter, because he saves his explanations until the last part of the last chapter of his book … and devotes over 50 pages of his manuscript to them.

Leonardo must have thought that the title of this third part needed to clearly state its content, precisely because his future readers would be unfamiliar with what it was. And that is why he chooses a Latin title incorporating two Arabic words, *algebra* and *almuchabala*, which he immediately explains. He translates the first as *proportion* and the second as *restoration*. He therefore transliterates only the first word of the expression that was used to designate algebra in the Arabic-speaking world at this time: *al-jabr wa l-muqabala*. The word *al-jabr* had several meanings during this period, including setting a fractured bone, which, when applied to calculation, could either mean transposing a subtracted term or making a comparison. And the term *al-muqabala* – which evoked the idea of comparison or opposition – had come to signify comparison in mathematical parlance. The full phrase therefore referred to restoring and comparing terms. But it quickly became common to use the abbreviated form – *al-jabr* – which gave us our modern word *algebra*.

Figure 6**: First page of the treatise* **al-mukhtaṣar fī ḥisāb al-jabr wa-l-muqābala by** ***al- Khwārizmī, c.813–830 *(WikiCommons)*

[EXTRACT 3] THREE ELEMENTS, SIX MODES

Leonardo makes a distinction between three algebraic elements: the "root", or what we would call the unknown, the *census*, which is the square of this unknown, and the "simple number", in other words the constant. It is lucky that Leonardo tells us that the word *census*, which is used for the first time here, means *square*, because in everyday Latin the word meant "quantity" or "fortune" (as in the original meaning of the word *census* in ancient Rome: "the registration of citizens and their property, for purposes of taxation"). Though we were tempted to translate the term as *square*, this would have led to confusion with the geometrical shape of the same name. We have therefore left it in Latin, at the risk of complicating the demonstration below!

The author now mentions the "solutions to the problems", which, given the context, must mean the equations representing these problems. Leonardo distinguishes six "modes", the first three of which are "simple". Expressed in modern mathematical languages, they are:

$$Ax^2 = Bx$$
$$Ax^2 = B$$
$$Ax = B$$

Leonardo gives a few examples, though we will cite only the first one here. In modern notation, "two *census* are equal to ten roots" would be written as:

$$2x^2 = 10x$$

Leonardo first divides this by 2 (which would give $x^2 = 5x$) and then observes, in a particularly impenetrable phrase, that his unknown equals 5. As one might expect, there is no mention of the solution $x = 0$, given that mathematicians of this era were interested only in positive values.

In another quirk of mathematical history, the x on this page – which makes *us* think of a variable – is for Leonardo nothing more than 10 written in Roman numerals. After several hundred pages in a book that is intended to showcase the richness of Arabic numerals, the author often slips into using Roman numerals in his text.

He then describes the three other modes, which he describes as "composite", followed by a few more simple examples, which, again, we have left out here:

$$Ax^2 + Bx = C$$
$$Ax + B = Cx^2$$
$$Ax^2 + B = Cx$$

It is worth noting that this classification into six modes is exactly the same as that found in the writings of al-Khwārizmī.[16] In addition, the operations included in *al-jabr wa l-muqabala*, which we would describe as first- and second-degree equations, always lead to one of these six types of equation.

[EXTRACT 3, CONT.] **BUT WHAT ALGEBRA!**

We now come to a trickier example, one which underlines the inefficiency and complexity of the Middle Eastern algebra that Leonardo describes. It is a short algebraic equation, $x^2 + 10x = 39$, whose solution "in algebra" is described in the first five lines of the third part of our extract:

> *For example, a census and ten roots equal 39. Half of the number of roots is 5, which multiplied by themselves give 25, which added to 39 give 64, of*

16. See Rashed (2007), *Al-Khwarizmi, le commencement de l'algèbre*, pp. 96–106.

which the root is 8; if one subtracts half of the number of roots from it, that is to say 5, one is left with a figure of 3 for the root of the census sought. Therefore the census is 9, and its ten roots are 30, and therefore the census and the ten roots equal 39.

Expressed in modern notation, what Leonardo is saying is:

$$10 : 2 = 5$$

$$5^2 = 25$$

$$39 + 25 = 64$$

$$\sqrt{64} = 8$$

$$8 - 5 = 3$$

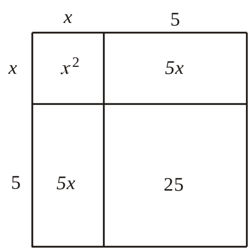
This produces the answer of 3, and, as one should expect, Leonardo makes no mention of the solution $x = -13$ because he is only interested in – indeed, he only knows about – strictly positive values. These operations are similar to what we now call completing the square, which would be written as:

$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25$$

$$(x + 5)^2 = 64 \mid x + 5 = 8 \mid x = 3$$

But what is even more characteristic about this passage is the fact that the author justifies his solution with *geometric* reasoning. This once again underlines the influence of Euclidean geometry on Leonardo and on Arabic science as a whole. In addition, the reasoning is so complex that it's enough to make the reader want to give up there and then![17] But once translated into modern notation, it becomes much easier to understand:



- We know that $x^2 + 5x + 5x = 39$
- and that the area of the square equals $x^2 + 10x + 25 = 64$
- so the side is equal to the root of 64, i.e. 8
- so $x + 5 = 8$, so $x = 3$.

17. In our translation, we have skipped a sentence (indicated by '[…]') because it is incomprehensible. Sigler also excludes it from his translation. It may have been an unwelcome addendum by a scribe.

[EXTRACT 4] LEONARDO ALSO INTRODUCED FALSE POSITION

Leonardo's essential contribution to the West was not limited to the new method of calculation or to algebra. Among his other important innovations, special attention should be paid to his presentation of the method known as "false position", which is a means of solving what we call first-degree equations without using algebra.

Basically, the method consists in 1) positing a *false* value as the solution to the problem, 2) "working out" the problem and observing that the solution has not produced the correct result, and 3) calculating the correct solution using proportionality. In many respects, this technique is the acme of the new method of calculation.

The method was known and used in ancient Egypt, as we have shown elsewhere.[18] In the West, it is highly likely that the method was already used in the time of Alcuin of York, circa 800, for some of the problems that author describes seem custom designed to use false position (but, unfortunately, we do not know how he solved them). Yet Leonardo was the first to present the method, demonstrating how it worked more fully than it had ever been before and ever would be afterwards. It is therefore worth going back to these pages in his book and examining the two extracts that illustrate this contribution.

[EXTRACT 4, CONT.] SIMPLE FALSE POSITION ...

There are two methods of false position: simple and double. But most authors presented only one of the methods, saying nothing about the other. Leonardo is rare in that he describes both, yet he does not consider simple false position as a method in itself: for him, it is a variation of the Rule of Three.

This can be seen in his first example. He first solves the problem (in the first line in our extract), presenting it as a standard application of this rule, which he has set out in the previous pages. Then, on the following line, he announces "another way" of solving the problem, which we would describe as false position: Leonardo chooses 12 "because it is easily divided by 3 and by 4" – in other words,

18. See Gavin and Schärlig (2012), *Longtemps avant l'algèbre, la fausse position*. See also the following article by the same authors "Schreiber alias Grammateus: From 'false position' to the tentative beginnings of algebra", BibNum, November 2014.

if the value is going to be incorrect, we might as well go for something simple. He then performs the calculation, the answer to which is 7. Yet since we were trying to obtain an answer of 21, 12 must therefore be three times too small, so the solution must be three times 12, that is to say 36.

It's worth pointing out in passing Leonardo's curious way of measuring the buried part of the tree: 1/4 1/3, which reads "a quarter and a third". Leonardo was aware that this represented 7/12 of the tree, as can be seen in his subsequent calculations. And he wouldn't have minded writing it like this, because in his book, both before and after this particular passage, one finds fractions with large numbers in the numerator. But he didn't wish to write it like this here. He was probably trying to avoid taking his reader aback at the very moment he was introducing his "tree problems" – in other words, what we would call simple false position. Indeed, in Leonardo's time, many people were not familiar with reciprocals (or unit fractions), i.e. fractions in which the numerator is the unit, as had been the case in ancient Egypt and ancient Greece.

[EXTRACT 5] ... AND THE TWO TYPES OF DOUBLE FALSE POSITION

False position is presented much later on in the text as a method in its own right,[19] known by its Arabic name *elchatayn*[20] (sometimes written as *elchataym*). It is used in apparently more complicated cases where there are *two* constants. For modern readers like ourselves, familiar with algebra, this added complexity is only superficial: all one needs to do is combine the two values to the right of the equal sign. But the mathematicians of previous centuries hadn't picked up on this. Instead they invented a method to take both values into account.

By virtue of the adage "He who can do more can do less" (*Qui peut le plus peut le moins*), it is acceptable to use the double false position to solve problems that require only simple false position. Leonardo turns this principle into a teaching tool: he starts his chapter with double false position, applying it to a problem where simple false position would suffice. This is the tree problem shown in the extract above, and it allows him to demonstrate double false position without saying so in so many words.

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19. Note that Leonardo, like later authors, uses the word *rule* where we would talk about a *method* or *technique*.
20. A word signifying "the two errors", which encapsulates what "double false position" is.

This is an opportunity for him to present the *two* variants of the method, something that later writers rarely did and which, as far as we are aware, is unheard of among earlier authors. As he writes in his text – and clearly this time! – the calculation is performed using the principle of proportionality, or by applying a formula. Hardly surprisingly, it is this second option that history has preferred![21]

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The reader will notice that Leonardo's description of the differences as either "augmentation" (*augmenti*) or "diminution" (*diminucionis*) is the same as that found in *The Nine Chapters*, the oldest Chinese work of mathematics that refers to false position (dating from the 2nd century BCE). Yet the latter does nothing more than describe the method used in the formula!

At the end of the second line, the reader may also notice another example of the influence of Arabic notation on Leonardo's thought. Because Arabic is read from right to left, Leonardo places the fraction to the right of the number, giving us 1/5 7 instead of 7 1/5!

The calculations that Leonardo describes in his example of the two methods are easy enough to follow. So we will simply point out that from the time of Charlemagne onwards, one French livre was equal to 20 sous, and one sou was equal to 12 deniers. It then becomes easy to understand why the 2 sous and 7 1/5 deniers in the first calculation become 2 3/5 sous in the second calculation: 3/5 of a sou are worth 12 times 3/5 deniers, or 36/5 deniers (that is, 7 1/5 deniers).

In conclusion

Leonardo da Pisa was one of the first people in Europe to describe Arabic numerals, but above all, he was the first to describe the written calculations they made possible. He described this new method of calculation in 1202, as it was then practised in the Arabic world, that is to say after four centuries of development (whereas Sacrobosco, writing around 1230, published a mere reflection of what the method had been when it originally emerged in Baghdad circa 850). Leonardo was perhaps not the first person in the West to teach false position – mathematical

21. Readers wishing to find out more about Leonardo's explanation will find the formula in modern notation, and a more detailed commentary, in our other article on BibNum.

problems presented by Alcuin, written around 800, were clearly designed to illustrate simple false position – but he was the first to describe it in a book and to describe it fully (which was also rare in subsequent decades and centuries). He was also the first to describe algebra, though, as we have seen, much progress remained to be made!

And, *incidentally*, he wrote a story about mating rabbits – just one problem among a host of others – in which a 19th-century mathematician recognised a *sequence*, which Leonardo hadn't noticed, but which would make him famous … as Fibonacci.

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