How Newton might have derived the Coriolis acceleration

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The 1905 edition of the German scientific journal *Annalen der Physik* has become famous for publishing Albert Einstein's five ground breaking articles which would open the door for "modern physics". It is less well-known that in the same 1905 edition of *Annalen* there was a heated debate between three Central European scientists[1] on the true understanding of the Coriolis Effect. When the three didn't reach any agreement, the Editor-in-Chief, Max Planck, asked them to continue their discussion somewhere else[2].

Today we are proud ourselves of having a fair understanding of quantum mechanics and relativity theory – but what still baffles our minds is the Coriolis Effect! No wonder that Alexandre Moatti gave his biography of Gaspard-Gustave Coriolis (1794-1843) the title *Le Mystère Coriolis*[3].

1. Coriolis's memoire of 1835

In the 1830s, with the industrial revolution in full swing, Coriolis became interested in the dynamics of machines with rotating parts. How much did the centrifugal force, and thus the strain on the machine, change when a part of the machine was also moving relative to the rotation?

In his 1835 mémoire Coriolis showed that in a machine that was rotating with an angular velocity Ω where a part was moving relative to the rotation with velocity V_r, one had to add to the ordinary centrifugal force Ω^2R or U^2/R (where R is the distance to the rotational axis and U the rotational speed) a supplementary force $2\Omega V_r$. This additional force, which much later became known as the "Coriolis

1. One German and two Austrians, who today would have been regarded as Polish, Czech and Ukrainian.
2. Which they did and turned to Physikalische Zeitschrift in 1906 until its Editor-In-Chief got tired of them as well!
3. His book (CNRS Editions, 2014) and his *BibNum* contribution (October 2011) contain a detailed review of Coriolis's 1835 mémoire "Sur les équations du mouvement relatif des systèmes de corps" Journal de l'École polytechnique, 24° cahier, XV, cahier XXIV, p. 142-154. It suffices here just to recapitulate the general idea.

force", was independent of R and perpendicular to the relative motion V_r. In a counter-clockwise rotation it was directed to the right (to the left in a clockwise). Because of the right angle deflection of the motion it could not change its speed (its kinetic energy), only the direction. Coriolis called this additional force "the composed centrifugal force".

It will be argued here that the problems over 180 years to understand this "Coriolis force" stems not only from mistaken approaches to conceptually consider it in isolation, independently of the centrifugal force, but also attempts to visualize it in a relative frame of reference. It is here that ideas from Isaac Newton's *Principia* may come to rescue.

2. Depicting fictitious forces

In the standard vector derivation (relative acceleration in a rotating system Ω of a relative motion V_r at a distance R), the two forces are organically linked to each other (see eq 1):

$$\left(\frac{dV_r}{dt}\right)_{rel} = \left(\frac{dV}{dt}\right)_{fix} - \Omega\times(\Omega\times r) - 2\Omega\times V_r \qquad (1)$$

Although the Coriolis term – $2\Omega\times Vr$ – appears to have a different mathematical structure than the centrifugal term $\Omega\times(\Omega\times R)$, they are physically of the same centrifugal nature[4].

2.1 An erroneous depiction of the centrifugal force

Another source of misconception is how we tend to erroneously visualize fictitious forces, not only the Coriolis force but also the centrifugal force. A popular way to depict the latter is by an image which contains the relevant parameters, the rotational velocity U, the distance to the centre of rotation R and the rotation Ω (here anticlockwise). It could look something like this, with the centrifugal force pointing away along a line passing through the centre of rotation (figure 1).

4. The centrifugal term can also be written as a product of the angular rotation (Ω) and a velocity, in this case the angular velocity ($U=\Omega\times R$).

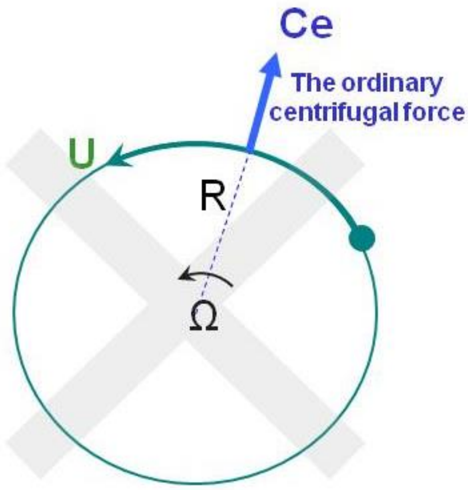


Figure 1: *A common popular, but misleading, image of the relation between an object in curved motion (U) and the centrifugal force (C_e) at distance R from the centre of rotation Ω. The image is crossed over because it is not quite correct.*

But doing so we are mixing two different frames of reference. The trajectory of the motion is displayed in an absolute frame of reference, as seen from outside. But the centrifugal force can only be experienced from "inside" and can only be depicted in a relative frame of reference.

Still the picture makes some "common sense" since, riding on a carousel or in a bus taking a curve, we can visually estimate the curvature. The forces we feel are "inside" the carousel or the bus, but our eyes reach "outside".

2.2 A correct but uninteresting depiction of the centrifugal force

But imagine that we are blindfolded – or travel by night in high-tech smoothly running train. Now it is almost impossible to find out in which direction the train is moving (backward or forward?). We notice when the train takes a curve only because the centrifugal force pushes us in one direction, and if we sit properly our seat provides an opposite directed centripetal force that keeps us in place. This is then what we will experience with our senses (figure 2):

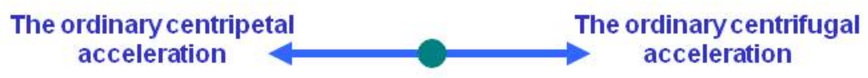


Figure 2: *The same motion as in figure 1 but now depicted only in a relative frame of motion, following the moving object, and not in an absolute frame of reference where it is seen from outside the object.*

This physically correct image might, however, appear a bit "dull" and "uninteresting". There is, for example, no information about the curvature of the motion or if it is forward or backward.

2.3 An erroneous depiction of the Coriolis force

If we still, stubbornly, insist on an erroneous, but "easily understood" image of the Coriolis deflection it could look something like this (figure 3):

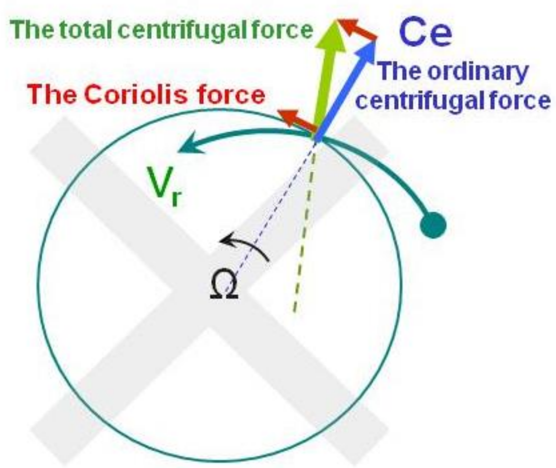


Figure 3: *An easily understood, but misleading, image of the Coriolis force as one of the two components of a decomposed centrifugal force.*

With an inward motion V_r the trajectory of the absolute motion is no longer exactly along the rotation but spirals radially inwards, towards the centre of rotation. The centrifugal action, perpendicular to the trajectory, is no longer pointing along a line passing through the centre of rotation.

We decompose the vector C_e into two vectors. One is passing straight through the centre of rotation and is the "ordinary" centrifugal force $\Omega^2 R$. The other is perpendicular and that is Coriolis' "composed centrifugal force" $2\Omega V_r$. Since the relative motion V_r is radial inwards, we can also see that this additional force is pointing to the right of the relative motion, for anticlockwise rotation (like the Earth's).

We are faced with a contradiction, on one hand an "understandable" but incorrect image, on the other hand a correct image which doesn't make much "common sense". But there is a solution – we abandon the relative frames of references and only use *absolute frames of reference.*

3. Depiction of forces in an absolute frame of reference

By a strange international convention the "Coriolis acceleration" is not the Coriolis force per mass unit, but an acceleration, linked to the centripetal acceleration, due to *a real force*, pointing in the opposite direction to the Coriolis force – just like the centripetal force is pointing in the opposite direction to the centrifugal force!

To start with the latter it is the action of an inward central force U^2/R (figure 4).

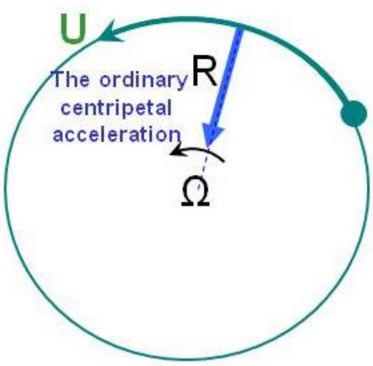


Figure 4: *A constant centripetal force of magnitude $\Omega^2 R$ drives an object into a circular motion. Instead of an* ***outward centrifugal force*** *because of the curved motion, we have a curved motion because of an* ***inward centripetal force****.*

The inward spiralling trajectory due to a tangential velocity U and a radial velocity V, can now be seen as the result of an inward centripetal force, which can be decomposed into one ordinary centripetal force, directed to the centre of rotation, and a perpendicular force, now orientated to the left of the relative radial motion, the Coriolis acceleration (figure 5).

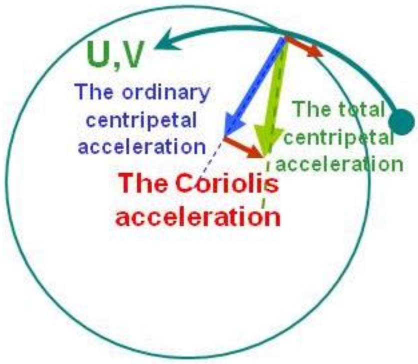


Figure 5: *To make the tangentially moving body also move radially inward a centripetal acceleration has to be applied, which is the sum of the ordinary centripetal acceleration and the Coriolis acceleration.*

It can also be said that the Coriolis acceleration has to be applied on a moving object to avoid it being deflected by the Coriolis force. To understand the Coriolis deflecting mechanism by studying the Coriolis acceleration in an absolute frame of reference may sound controversial, but we have a powerful supporter in Sir Isaac Newton, author of *Principia*!

4. The use of geometry in Newton's Principia

The world famous *Principia* is available in translations in most languages. Still, few of us have read the book, or even opened it. This is a pity because just glancing through the pages gives a surprising revelation: there are no algebraic equations, only Euclidian geometry. By using Euclidian geometry, Newton thought that his results would be more easily understood, at least in his time (figure 6a).

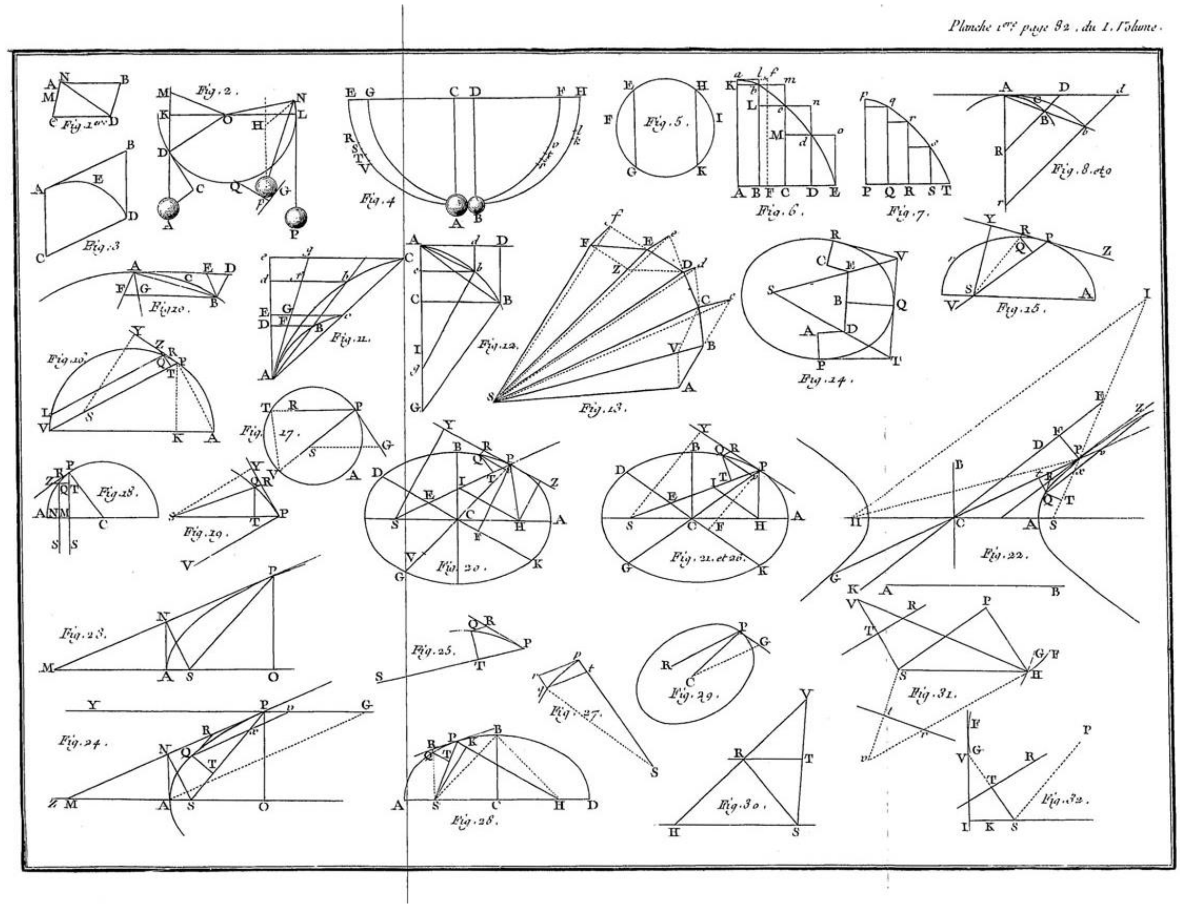


Figure 6a: *A map from the French edition of* Principia *displaying all of the figures in Part I. The figure we will be particularly interested in is "Fig.13" in the middle.*

We will give an example of Newton's Euclidian approach that is related to the problem of calculating the centripetal force. It also illustrates the mathematical beauty of his reasoning.

SECT. II.

De Inventione Virium Centripetarum.

Prop. I. Theorema. I.

Areas quas corpora in gyros acta radiis ad immobile centrum virium ductis deſcribunt, & in planis immobilibus conſiſtere, & eſſe temporibus proportionales.

Dividatur tempus in partes æquales, & prima temporis parte deſcribat corpus vi inſita rectam *AB*. Idem ſecunda temporis parte, ſi nil impediret, recta pergeret ad *c*, (per Leg. I) deſcribens lineam *Bc* æqualem ipſi *A*B, adeo ut radiis *AS*, B*S*, *cS* ad centrum actis, confectæ forent æquales areæ *A S*B, B*Sc*. Verum ubi corpus venit ad B, agat vis centripeta impulſu unico ſed magno, faciatq; corpus a recta B*c* deflectere & pergere in recta B*C*. Ipſi B*S* parallela agatur *c C* occurrens B*C* in *C*, & completa ſecunda temporis parte, corpus (per Legum Corol. 1) reperietur in *C*, in eodem plano cum triangulo A*SB*. Junge *SC*, & triangulum *SBC*, ob parallelas S*B*, *Cc*, æquale erit triangulo *SBc*, atq; adeo etiam triangulo *SAB*. Simili argumento ſi vis

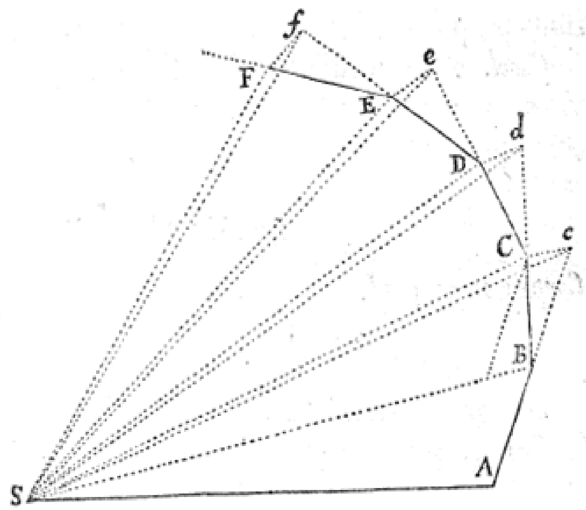
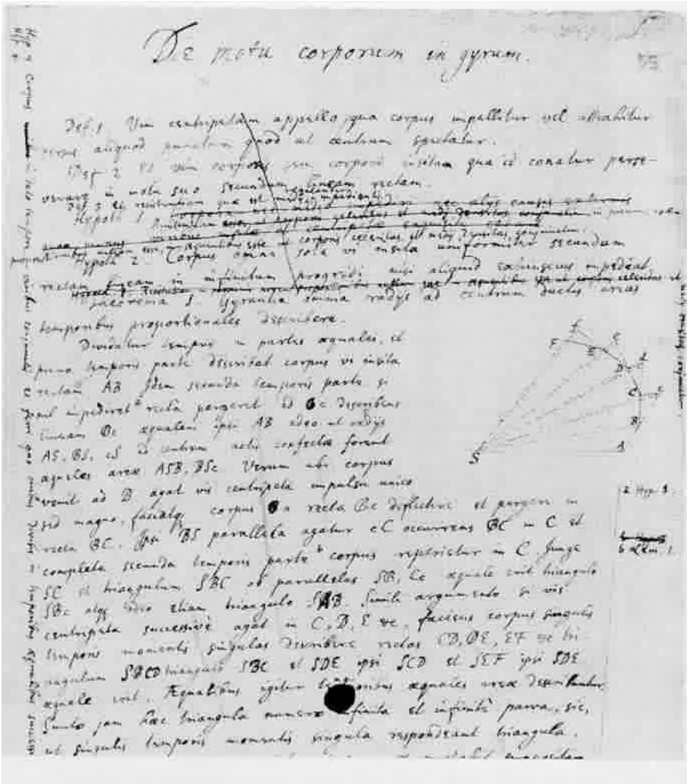


Figure 6b: *One of the pages in* Principia *with the geometrical construction Isaac Newton used to prove Kepler's Second Law, which is the same as conservation of angular momentum.*



De motu corporum in gyrum.

Figure 6c: *Before* Principia*, Newton made a shorter text called "De motu corporum in gyrum", where he tested some of his ideas.*

5. Applying Newton's method

Let a body move without friction rectilinearly from A to c over B, B being the middle of the segment (figure 7).

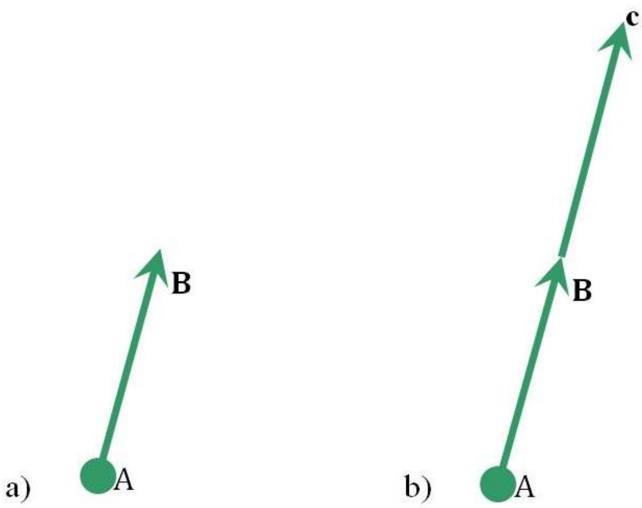


Figure 7: *A body is moving without friction rectilinearly from A to B (a) and then from B to C (b). In terms of distance, AB = Bc. Newton's notations are used.*

Newton showed early on in *Principia* that the radius vector of a body moving rectilinearly under inertia during equal times covers equal areas and thus obeys Kepler's Second Law (Figure 8).

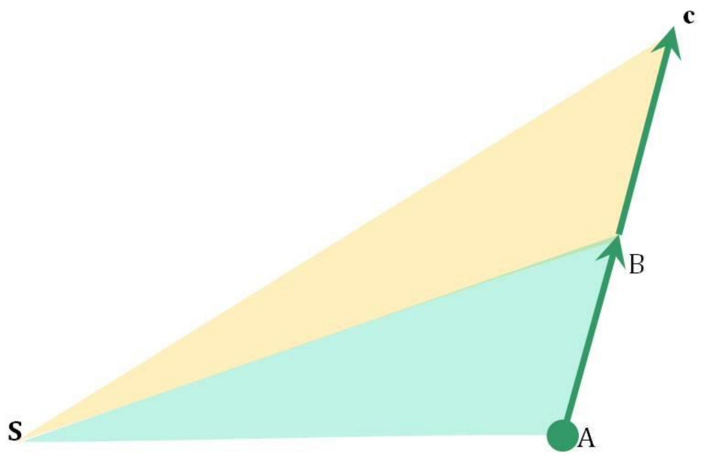


Figure 8: *The radius vector of the body's motion in relation to point S covers in equal times equal areas (green and yellow) in accordance with Kepler's second law.*

Newton assumed that the body at B was subject to an impulsive force which, had it been alone, would have moved the body in a new direction (figure 9a). But in combination with the original motion the body will move in a combination of the two directions (Figure 9b).

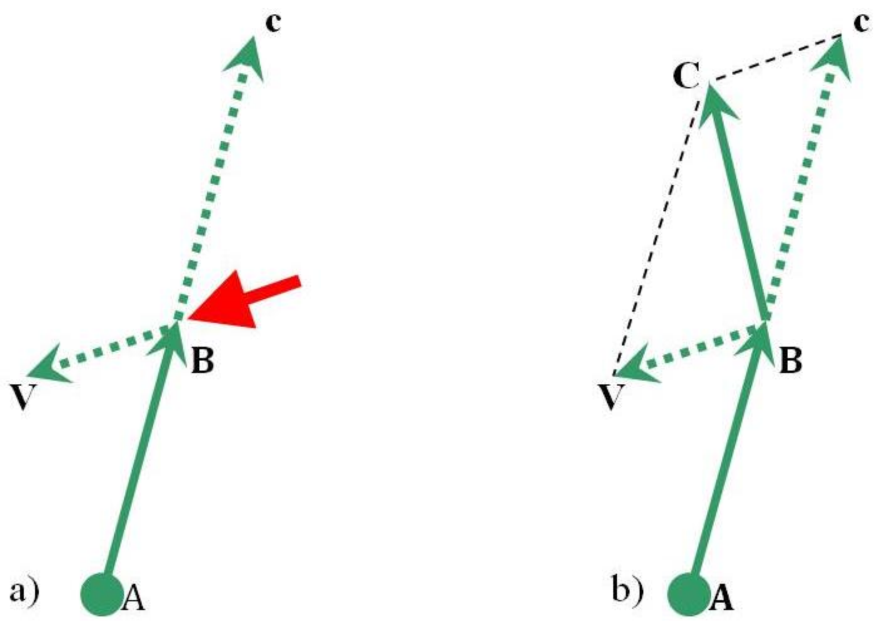


Figure 9: *An impulsive force pushes the body at B in the direction of V (a). The resulting motion to C is a combination of the motions from B to V and B to c (b).*

From figure 8 we saw that the green and yellow triangles have equal areas. But what about the new red-dashed triangle SBC in figure 10?

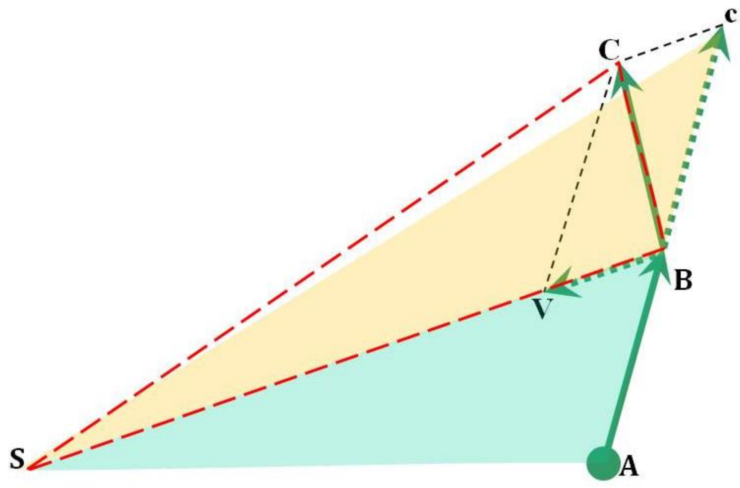


Figure 10: *The area BSC covered by radius vector during the motion from B to C (red dashed line) will be shown to have the same area as the yellow area BSc covered by radius vector during the motion from B to c.*

Newton showed that since both have the same base SB, and the same height CC′ (equal to cc′), their areas are equal (figure 11).

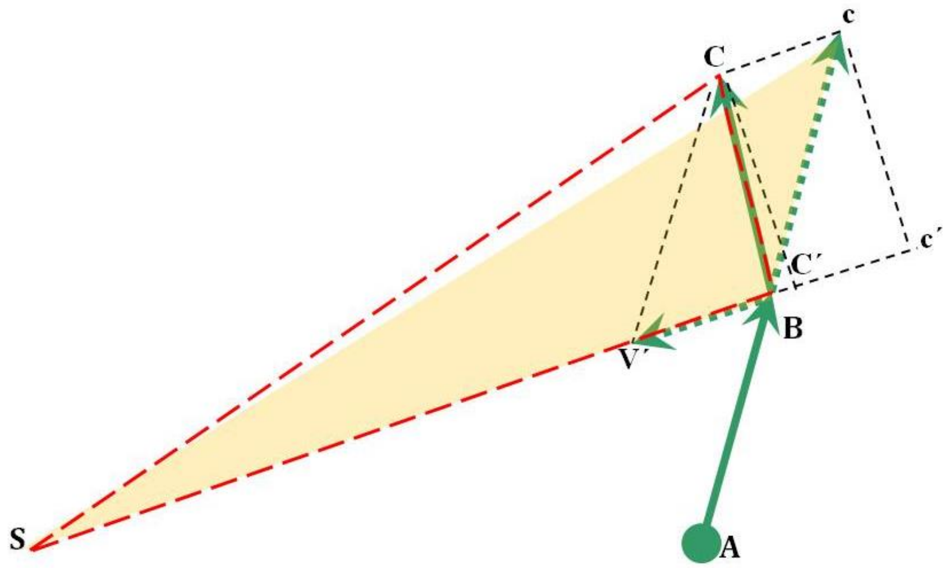


Figure 11: *Newton's proof that the areas BSC = BSc*

Although the body has deviated from its rectilinear motion into a curved motion, it still fulfils Kepler's Second law by having its radius vector cover equal areas in equal times (figure 12).

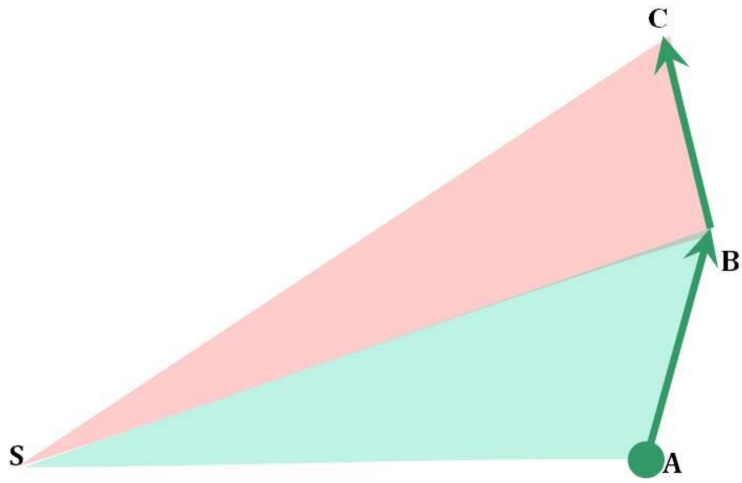


Figure 12: *The body moving from A to C over B has its radii vectors cover equal areas in equal times according to Kepler's Second Law.*

Newton then applied the same reasoning for subsequent time steps: in every point C, D, E and F the body was affected by an impulsive force pointing to the same central point C (figure 13).

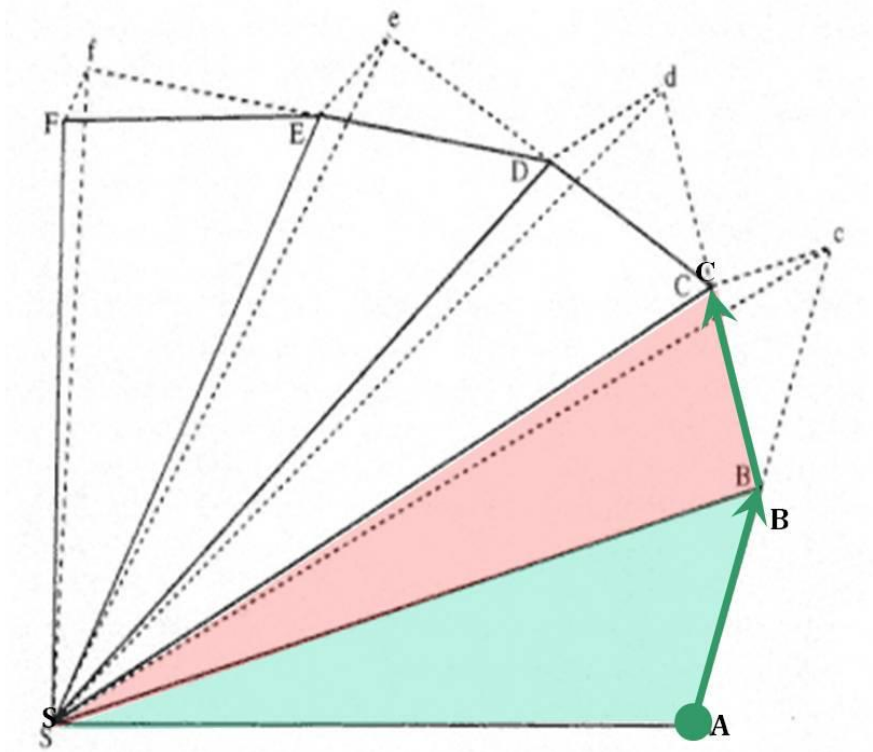


Figure 13: *Figure 12 inserted in its context in Newton's* Principia*.*

Kepler's Second Law, also called 'The Area Law' was in the 1600s and most of the 1700s thought to apply only to celestial objects (planets, moons, comets etc). But already in *Principia*, Newton showed that the same laws of motion which guide the heavenly objects also are active on Earth, among them the Area's Law, which we today call "conservation of angular momentum".

6. Newton and the centripetal force

Long before Newton thought about deriving Kepler's Second Law, he was interested in calculating the centripetal force. The centrifugal force had been calculated by Huygens in 1659 and Newton's first attempts dates from before 1669 (when he was less than 27 years old).

It is quite fascinating how he used the little-known Proposition 36 from Book III of Euclid's *Elements*. Here Euclid proved that a rectangle constructed over the diameter CE and the short distance CD has the same area as the quadrate defined by the length BC (figure 14a).

Thus CE × CD = $(BC)^2$. We now apply it on for our purpose more relevant symbols in figure 14b).

$$2r \cdot \Delta r = (u \cdot \Delta t)^2 \tag{2a}$$

$$\Delta r = \frac{u^2 \cdot (\Delta t)^2}{2r} = \frac{u^2}{r} \cdot \frac{(\Delta t)^2}{2} \tag{2b}$$

$$acc = \frac{u^2}{r} \tag{2c}$$

which is the equation for the centripetal force (per mass unit).

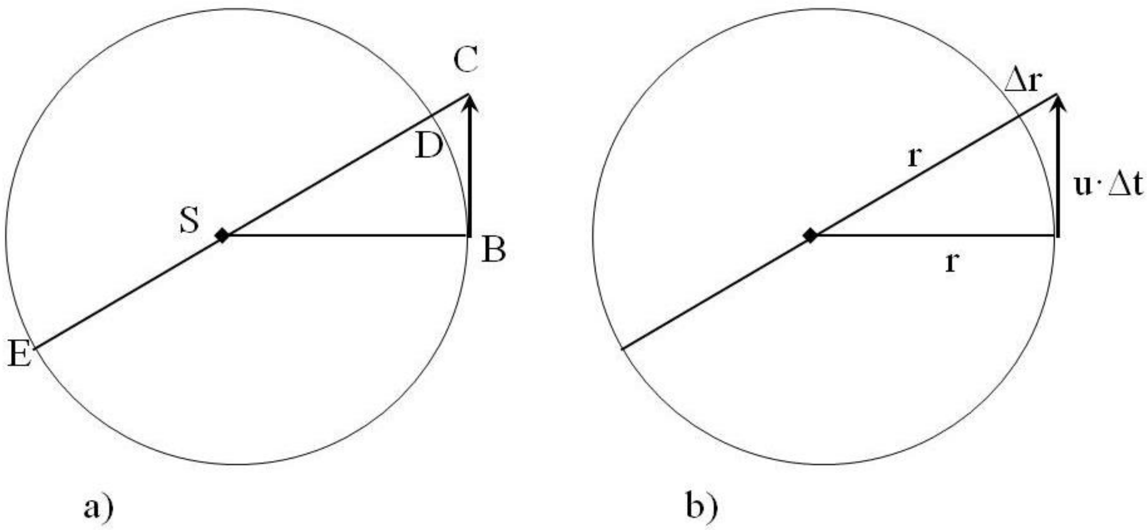


Figure 14: *How Isaac Newton derived the centripetal force from Euclid's proposition 36 in Book III with his notations to the left (14 a), the ones applicable to this article to the right (figure 14 b).*

Newton's *Principia* might have been a relatively "easy read" 300 years ago, but for a modern reader the geometrical language is not immediately comprehensible. On the other hand, any effort seems to be highly rewarded by these beautiful Euclidian proofs!

7. Some Coriolis derivations à la Newton

Inspired by Newton we will now apply a geometrial approach to derive the centripetal acceleration and the Coriolis acceleration for tangential and radial motions in an absolute frame of reference.

7.1 Tangential motion

We will first derive an expression for the centripetal force, using more or less the same approach as Isaac Newton in his pre-1669 paper (figure 15).

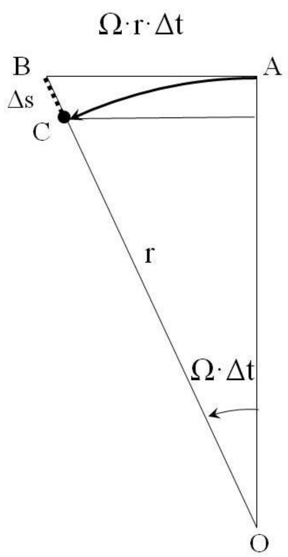


Figure 15: *A geometrial derivation of the centripetal force.*

A body is carried around with a rotation **Ω** at a constant distance *r* from the centre of rotation O. An inward directed centripetal acceleration **a** makes the body deflect an inward distance Δs over time Δt. Without this inward force the object would move rectilinearly from A to B under inertia $\Omega \cdot r \cdot \Delta t$. Following Pythagoras theorem applied to ABO triangle, we have $(\Omega \cdot r \cdot \Delta t)^2 + r^2 = (\Delta s + r)^2$ which yields, by neglecting second order terms, $\Delta s = (\Omega \cdot r \cdot \Delta t)^2/2r = a \cdot (\Delta t)^2/2$, which results in **a** = $\Omega^2 r$, the centripetal acceleration.

To the tangential motion ΩR is now added a tangential velocity **u** (figure 16).

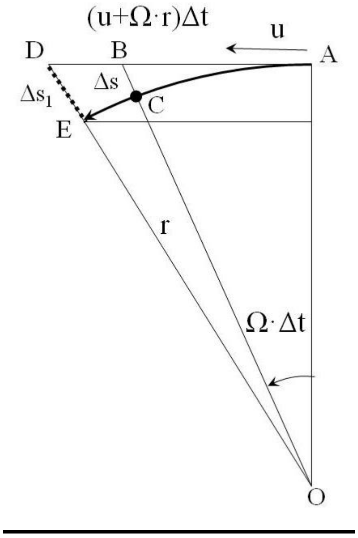


Figure 16: *Derivation of the centripetal force (per unit mass) for a relatively tangentially moving body with a speed different from the speed of rotation (see test for details).*

The centripetal force is as above: the tangential motion *u* is added to $\Omega \cdot r$

$$((\Omega \cdot r + u)\Delta t)^2 + r^2 = (\Delta s_1 + r)^2 \qquad (3a)$$

which yields after neglecting quadratic terms:

$$(u \cdot \Delta t)^2 + 2u.r.\Omega \cdot \Delta t^2 + (\Omega \cdot r \cdot \Delta t)^2 = 2\,\Delta s_1 \cdot r \qquad (3b)$$

$$\Delta s_1 = (\Omega^2 \cdot r + 2\Omega \cdot u + u^2/r)\,\Delta t^2/2 \qquad (3c)$$

i.e. the centripetal acceleration $\Omega^2 \cdot r$, the Coriolis acceleration $2\Omega \cdot u$ and a so-called "metric acceleration term" u^2/r. This term turns up in the simplest of the Coriolis force derivations (see below) and we will now for a short while return to the relative frame of references where centrifugal forces exist.

7.2 "Metric" terms

The centrifugal force (per unit mass) (C_e) on an object at distance *R* from the centre of rotation is $C_e = U^2/R$, where $U = \Omega R$ is the tangential rotational speed. In the case of a relative tangential motion ur, the total centrifugal force becomes

$$\frac{(U+u_r)^2}{R} = \frac{(\Omega R+u_r)^2}{R} = \Omega^2 R + 2\Omega u_r + \frac{{u_r}^2}{R} \qquad (4)$$

where the first term is the ordinary centrifugal force $\Omega^2 R$ and the second term the Coriolis force $2\Omega u_r$. The last metric term u_r^2/R is often explained as a reflection of the chosen coordinate system[5]. For normal velocities it is always weaker than the Coriolis term. They are equal when

$$2\Omega \cdot u_r = \frac{{u_r}^2}{R} \rightarrow u_r = 2U \qquad (5)$$

where U is the speed of rotation ΩR.

In the geophysical sciences, where R refers to the radius of the Earth, the metric term is often disregarded since $u_r^2/R \ll 2\Omega\cdot u_r$ when u_r does not exceed 100 m/s. But this mathematical-numerical argument misses the physical reason why the metric term can be disregarded in many practical applications. Although it does not contain Ω, and therefore is not related to the rotation, it is nevertheless not without a physical meaning. We may note its "centrifugal nature", i.e. a squared velocity divided by a distance. This reminds us that centrifugal forces do not only appear with rotation, but any curved motion.

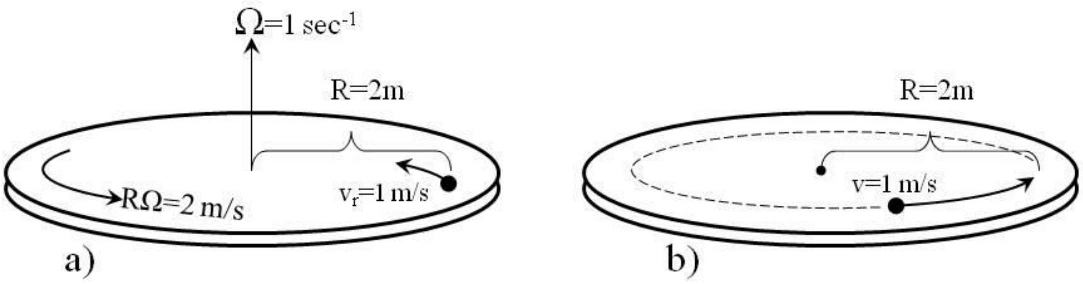


Figure 17: *Free relative motion v_r over a rotating platform (a) and constrained curved motion v over a non-rotating platform (b).*

In figure 17a (left) a body is moving freely over a rotating platform (e.g. a carousel) and is affected by an outward centrifugal force of $\Omega^2 R$ = 2 m/s² and a Coriolis force $2\Omega v_r$ = 2 m/s². In figure 17b the platform is stationary but the body's motion is constrained, like in equation (2) above, perhaps by some rail (e.g. a children's toy train). The centrifugal force here is not due to any rotation, but is due to the curved motion.

7.3 The metric terms in geophysics

The physical rational for discarding the metric terms is not because $u_r^2/R \ll 2\Omega\cdot u_r$ but because the motions of the oceans currents and atmospheric winds are

5. Other metric terms might appear in a three dimensional spherical coordinate system where, with R in the denominator, the numerator can take the form v^2, uv, w^2, uw and vw, where v=radial motion and w=vertical. This is because each of the three equations, one for each dimension, expresses constrained motions along a latitude, longitude or in the vertical. When the three dimensional derivation is made in vectorial form the metric terms do not appear.

not constrained. However, a TGV train between Rennes and Paris is constrained to follow the rails and will experience a sideways acceleration both due the Coriolis term and the metric term (figure 18).

Figure 18: *A TGV train is running on a straight rail from Rennes to Paris at 47-48° latitude. With a speed of u_r = 90 m/s and at a distance from the Earth rotational axis of 4262 kilometers it is affected by two southward accelerations together 8.2 mm/s². One is due to the Coriolis effect (6.79 mm/s²) and the other to the curvature of the latitude (1.43 mm/s²). The latter acceleration, always directed towards the equator[6], would be present also on a non-rotating Earth. If the train had run in the opposite direction, from east to west, the Coriolis force 2Ωsin(latitude)u_r per mass unit would be directed in the opposite direction, towards the north and the total acceleration, now northward, would be 5.36 mm/s².*

An ice hockey puck gliding frictionless over the ice doesn't seem to be on a constrained motion, but it is, like most other bodies on our Earth, constrained by gravity and the solid surface's reaction force to remain on the Earth's surface. Moving with a speed V_r it will everywhere on the Earth experience a vertical upward acceleration

$$a = \frac{{V_r}^2}{R} \qquad (6)$$

which added to the vertical Coriolis effect accounts for the so called "Eötvös effect", which makes east-moving bodies lighter and west-moving bodies heavier[7]. Geodists have to take this into account when they measure the Earth's gravity from a moving platform. Now back to the "Coriolis force" and accelerations in an absolute frame of reference.

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6. The accelerations due to both the Coriolis term and the metric term are directed perpendicular to the Earth's axis, and allowance has here been made for their local, horizontal component by a multiplication of the cosine of latitude.
7. In contrast to the metric term, the vertical Coriolis effect 2Ω$u_r$cos(latitude) is only dependent on the east-west motion u_r.

7.4 Radial motion

A body is moving with the rotation *Ω* from A to C, while simultaneously moving radially inward to G with velocity *v*, covering the distance $\Delta r = v \cdot \Delta t$. (figure 19a)

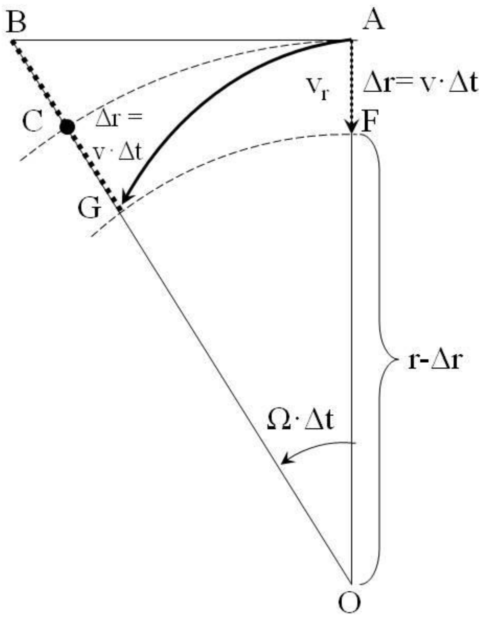

Figure 19a: *A body moving from A is taking part in the rotation while at the same time moving radially inward which leads to an arrival at G instead of C.*

Without inward motion the body would cover an arc $AC = r \cdot \Omega \cdot \Delta t \approx HC$ (figure 19b).

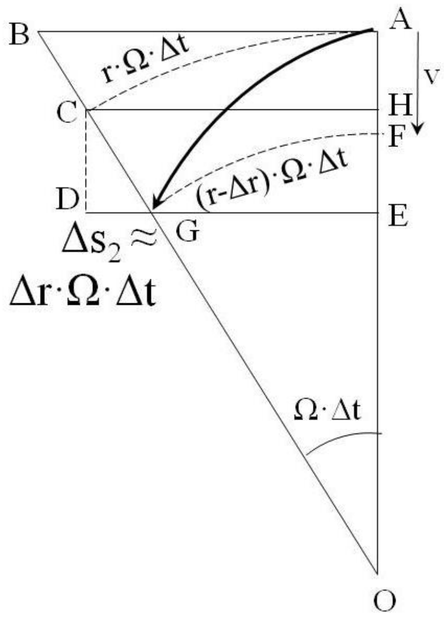

Figure 19b: *The calculation of the deflected distance* Δs_2

Using infinitesimal calculus allows us to regard the arcs AC ≈ HC and FG ≈ EG. With a deflection Δs_2 we have EG + Δs_2 = HC which yields Δs_2 =HC − EG ≈ AC − FG = $r\cdot\Omega\cdot\Delta t - (r - \Delta r)\cdot\Omega\cdot\Delta t = \Delta r\cdot\Omega\cdot\Delta t = a\cdot\Delta t^2/2$, with $a = 2\Omega\cdot\Delta r/\Delta t = 2\Omega\cdot v$ as the left directed acceleration.

8. Kepler's Second Law again

Finally, let's go back to where we started, with Isaac Newton's geometrical derivation in *Principia* of Kepler's Second Law, "loi des aires", or conservation of angular momentum. Denoted **L** angular momentum is the product of the angular velocity $\mathbf{R} \times \mathbf{\Omega}$ and the distance to the rotational centre **R**

$$\mathbf{L} = \mathbf{R} \times (\mathbf{R} \times \mathbf{\Omega})$$

(per unit mass) where **R** is the distance to the centre of rotation **Ω**. **L** and can often be treated as a scalar (L) since the rotational axis and the arm or lever are perpendicular to each other (as with spinning tops, ice skaters, ballet dancers and carousels). Then

$$L = R^2\Omega$$

Angular momentum is conserved if there is no torque, no force acting in the direction of the rotation. In the case above there is indeed a force acting counter to the rotation, the force that creates the Coriolis acceleration. Continuing to approximate the arcs with straight lines, we can see that in figure 17a the area EGO is smaller than the area HCO, meaning that the angular momentum has not been conserved but has decreased (figure 20a).

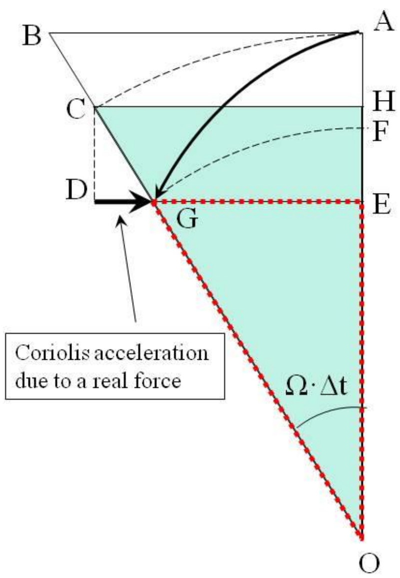

Figure 20a: *The area covered by the inward motion (red triangle) is smaller than the area covered by unaffected rotational movement (green area).*

However, with no torque the motion continues to D and the area covered by the motion has expanded and we can, with the same argument as Newton, easily prove that area EDO = area HCO and angular momentum is conserved (figure 20b).

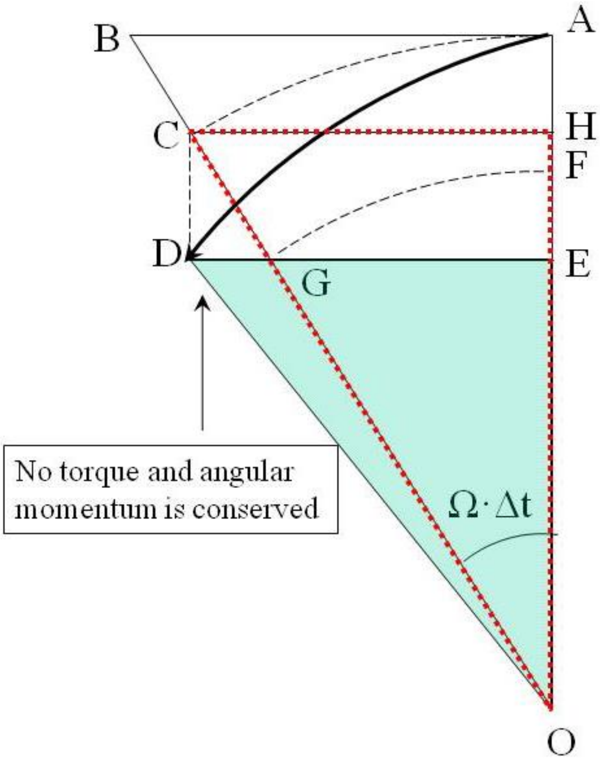

Fig 20 b: *The same as figure 17a but with no torque acting angular momentum is conserved.*

In the above example the angular velocity has increased. This agrees with familiar observations of spinning ballet dancers or ice skaters contracting their arms and spinning faster. The increase in kinetic energy derives from the work the ballet dancer or ice skater does with their muscles against the centrifugal force. So just because angular momentum is conserved, the rotational kinetic energy is not[8].

9. The beauty with mathematics

Newton's *Principia* and in particular his proof of Kepler's Second Law has fascinated many physicists, among them Richard Feynman (1918-1988) who used

8. This by the way goes back to a mémoire by Coriolis (1831), « Mémoire sur le principe des forces vives dans les mouvements relatifs des Machines », see BibNum.

it in chapter 2 of his book *The Character of Physical Law* (Penguin, 1964), among other examples, to clarify the relation of mathematics to physics:

> *Another thing, a very strange one, that is interesting in the relation of mathematics to physics, is the fact that by mathematical arguments you can show that it is possible to start from many apparently different starting points, and yet come to the same thing. (p.50)*

He then showed how the law of gravitation could be formulated in three seemingly different ways: as Newton's Law, as the local field method and as a variational "minimum principle".

> *They are equivalent scientifically... But psychologically they are very different in two ways. First, philosophically you like them or do not like them. Second, psychologically they are different because they are completely inequivalent when you are trying to guess new laws. (p. 53)*

The same is true for the mathematical derivations of the Coriolis effect: they are different, but yield the same result. You are free to choose the one you like and, at every instant, the one that helps you to draw new conclusions or even, as Feynman puts it: "guess new laws".

■

(July 2017)