Liouville, the discoverer of transcendental numbers

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In two 1844 entries to the *Comptes rendus de l'Académie des Sciences*, Joseph Liouville established the existence of transcendental numbers. What is a transcendental number? A number *x* is said to be algebraic if it is a solution of the following type of polynomial equation:

$$f(x) = ax^n + bx^{n-1} + ... + gx + h$$

Where *a, b,…., g, h* are non zero integers. Thus, for example, the following numbers are algebraic:

$$-5,\ 7/3,\ \sqrt{10},\ \sqrt[3]{5-\sqrt{2}},\ \sqrt{-1}$$

If the polynomial cannot be divided into two polynomials, *n* is called the degree of the algebraic number. The numbers of algebraic degree 1 correspond exactly to the rational numbers. The other algebraic numbers are called irrational: in the list above the respective degrees are 1, 1, 2, 6, 2.

Before Liouville, we could have believed that all numbers were algebraic. After him, we knew that there existed others: a non-algebraic number is said to be transcendental.

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Liouville presents two pieces of proof for the existence of such numbers; both rely on the theory of continuous fractions in order to establish the following fundamental:

If X is a real algebraic number with a degree of $n \geq 2$ *(therefore not rational), then there exists a non-zero positive constant C such that for each rational number p/q, we have:* $\left| x - \frac{p}{q} \right| > \frac{C}{q^n}$

In other words, even if the set of rational numbers is dense, a non-rational algebraic number cannot be sufficiently approximated with a rational.

Liouville's inequality can establish itself as the elementary means without the need for the continuous fractions that Liouville uses: this is why we will diverge from this part of his text, nonetheless ending up with the same result.

Here is one simple proof of this inequality. The definition of a derivative via difference quotient theory – well known to all first year university mathematics students – affirms that for every continuous and derivable real function *f,* there exists of a number *t* understood to be between *x* and *p/q* such that:

$$f(x) - f\left(\frac{p}{q}\right) = \left(x - \frac{p}{q}\right) f'(t)$$

If *x* is therefore algebraically real, there exists a polynomial with non-zero coefficients so that f(x)=0, the preceding inequality implying:

$$\left| x - \frac{p}{q} \right| = \left| \frac{f\left(\frac{p}{q}\right)}{f'(t)} \right| = \left| \frac{q^n f\left(\frac{p}{q}\right)}{q^n f'(t)} \right| .$$

We will look now to lower bound this quantity by one of the type A/q^n (Liouville's inequality).

Demonstration of Liouville's inequality

We will look firstly to lower bound the numerator q^n f(p/q). As Liouville had noted, this quantity, which he called f(p,q), is equal to $ap^n + bp^{n-1} +\ldots + hq^n$: it is a whole number. Liouville had taken the precaution to indicate that the polynomial *f* had been "cleared of any commensurable factor", that is to say, reduced to a form where it will allow only irrational solutions. Considering this precaution, the numerator can never be 0; acting as a whole number (positive or negative), its absolute value is always under bound by 1.

We will look now to over bound the quantity |f'(t)|, representing the denominator, by recalling that t is understood to be between p/q and x. We choose p/q so that – 1 < x – p/q < 1. So, with *f* being a polynomial function limited on the interval [x – 1, x + 1], it takes at this section finite values which we can over bound by an invariable quantity of 1 : |f'(t)| < A.

In documenting the above, after having under bound the denominator (with C = 1/A), we obtain Liouville's inequality:

$$\left| x - \frac{p}{q} \right| = \left| \frac{f\left(\frac{p}{q}\right)}{f'(t)} \right| = \left| \frac{q^n f\left(\frac{p}{q}\right)}{q^n f'(t)} \right| > \frac{C}{q^n}$$

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Once this inequality has been established, Liouville quickly mentions (in a sentence) the means with which he constructs a non-algebraic number (*"We quote in particular the series…").* Let us now detail the second part of his 'discovery'. He observes that the number

$$y = \sum_{k=1}^{\infty} \frac{1}{10^{k!}}$$

(We take *a* = *10* in the example that he gives at the end of the article)

is "too badly approximated" by the partial sum, $\sum_{k=1}^{N} \frac{1}{10^{k!}} = \frac{p_N}{10^{N!}} = \frac{p_N}{q_N}$, from which he concludes that it is a transcendental number. We will now detail the sum.

Liouville's series is a transcendental number

Let us address the quantity $y - \sum_{k=1}^{N} \frac{1}{10^{k!}} = \sum_{N+1}^{\infty} \frac{1}{10^{k!}} =$ *0,0000…….01……..*:

The first *1* appears in the position (N+1)! after the comma, and the other *1s* appear more and more spaced apart. We can therefore overbound this quantity. For illustration we will do so with the number where *2* appears in the position (N+1)! after the comma following the *0*:

$$\sum_{N+1}^{\infty} \frac{1}{10^{k!}} < \frac{2}{10^{(N+1)!}} = \frac{2}{10^{N!\,(N+1)}} = \frac{2}{(q_N)^{N+1}}$$

For every non-zero positive quantity of *C*, this last quantity will be overbound by $\frac{C}{(q_N)^n}$ when *N* is large enough that $q_N^{(N+1-n)} > \frac{2}{C}$ (a reminder that $q_N = 10^{N!}$). Therefore, for any given *n* and for any non-zero positive quantity of *c* we will be able to find an infinity of p/q (in the knowledge that p_N/q_N are sufficiently large enough for *n*) so that $\left| y - \frac{p}{q} \right| < \frac{C}{q^n}$.

This contradicts Liouville's inequality and allows for the conclusion that it is transcendental. We will observe in the passage that an irrational algebraic number cannot be closely approximated (Liouville's inequality), yet on the other hand certain transcendental numbers can be.

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Several years later, George Cantor demonstrated that "almost all" numbers are transcendental, which is quite surprising since by this point we are so much better familiarized with the zeros of polynomial equations! In 1873, Charles Hermite established the transcendence of *e*, then, continuing with similar ideas, Ferdinand Lindemann demonstrated in 1882 that π is transcendental. This last result solves once and for all the old problem of the squaring of a circle! We are convinced in effect that from the length of the unit, that all construction with the rule and compass can give only algebraic numbers (moreover, of the degree 1,2,4,8,16 etc. – Wantzel's theorem). If squaring the circle is possible, π would be algebraic, which is absurd.

Since, many families of transcendental numbers have been updated. If $a \neq 0$ or 1, and if *b* is irrational, both algebraic, so a^b is transcendental (A.O. Gelfond and Th. Schneider, 1934). In particular, since $e^\pi = (-1)^{(-i)}$, one sees that e^π is transcendental. In 1955, K.F.Roth improved the Liouville's inequality and showed that for every algebraically transcendental *x* and for every ε there exists an infinite number of irreducible *p/q* such that $|x- p/q| > 1/q^{2+\varepsilon}$. For this he was awarded the prestigious Fields medal in 1958.

This theory of transcendental numbers flourishes still today, without a doubt thanks to the fantastic work of mathematician Alan Baker during the 60s and 70s, for which he too was awarded the Fields medal in 1970.

The moral of this story is, it would seem to me, that Liouville's two memoranda, however perfectly elementary, are greatly profound. Only a great mathematician could discover ideas of such simplicity yet so greatly rich.

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