Niels Abel and Convergence Criteria

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***Figure 1:* A Norwegian bank note (1978) bearing the effigy of Niels Henryk Abel (1802-1829)**

The name Niels Henrik Abel may not allude to much, or else only to the eponymous prize created in 2001 by the Norwegian Academy of Science and Letters to make up for the absence of the Mathematics Nobel Prize. And yet, in popular imagination, Abel's romantic fate made it possible for him to compete with his famous contemporary Évariste Galois (1811-1832). Born in 1802, in a Norway stifled by the Napoleonic blockade, he proved to be a very studious pupil from a modest family. The return of a strict teacher (whose aggressive teaching would even go on to kill a pupil) and the arrival of a new and more open teacher helped the young Niels to thrive and become familiar with early mathematical success. The latter teacher, Bernt Holmböe, very quickly understood the ability of his pupil, as seen with the following assessment (which served as a recommendation) from which, as some people are to believe, we get a premonition of his tragic fate:

To the excellence of his intelligence unites a passion and an insatiable interest for mathematics, so much so that, without a doubt, if ever given the chance, he will like become a great mathematician.

After some early articles[1] in the journal *Magazin for Naturvidenskaberne*[2], Abel wrote, in 1824, a small mathematical wonder titled *Mémoire sur les équations algébriques oú l'on démontre l'impossibilité de la résolution de l'équation générale du cinquième degré [Thesis on Algebraic Equations in which the Impossibility of Resolving General Fifth-Degree Equations is Demonstrated]*. Thanks to this critical success, he won a scholarship from the Norwegian government for a study trip abroad (first of all in Germany and secondly in France) to meet renowned specialists, bask in their knowledge and also gain recognition for his own works.

The first few stops in Germany (Hamburg, Berlin, Freiburg) were quite promising, especially with meeting August Leopold Crelle who soon became a friend and an unconditional support for the young Norwegian. Unfortunately, due to a misunderstanding, Abel did not make a detour to Göttingen to meet with Gauss (who, in any case, no longer seemed interested in works on algebraic equations).

Figure 2:* *August Leopold Crelle (1780-1855), founder of **Journal für die reine und angewandte Mathematik** *(also widely known under the name of* Crelle's Journal*)*

He therefore hit the road to Paris since that is where, among others, the great Augustin Louis Cauchy (1789-1857) worked, the only one who understood and

[1] 1. His first article is titled *Méthode générale pour trouver des fonctions d'une seule quantité variable* [*General Method for Finding Functions of One Variable Quantity*], where a property of these functions is expressed by an equation between two variables and appeared in the first issue of the first volume in 1823.

[2] . The journal was cofounded by physicist Christopher Hansteen, professor of astronomy at the University of Christiana, who supported the young Niels Abel.

recognised the genius of the Norwegian. So full of hope was the young Abel (then only 23 years old) when he arrived in Paris and settled modestly in the Latin Quarter. It was not easy for the student to get in touch with the great scientists; with the Scandinavian's own shyness and the remoteness of luminaries, really more than anything this is about a missed rendezvous. Abel still tried to get himself known but ultimately had to admit his disillusionment. Yet, he was not mathematically discouraged, writing, in 1826, a long text titled *Mémoire sur une propriété générale d'une classe très étendue de fonctions transcendantes* [*Thesis on a General Property of a Very Extensive Class of Transcendental Functions*] for which he immediately filed a patent to the *Académie des sciences*. The rapporteurs were Adrien-Marie Legendre (1752-1833) and, rightly so, Cauchy. However, little news came back to him about this text and he recorded his despair in his correspondences, for example, in the letter to Holmboe[3], his former teacher:

> *Legendre is an extremely complacent man, but, unfortunately, also very old. Cauchy is mad, and with him there really is no way of getting along, although for now it is he who knows how mathematics should be treated. What he does is excellent, but very confusing.*

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Here is the tragic part of the story: Cauchy forgot or lost the manuscript[4] and as a result, Abel did not get the recognition he came to find. He still persisted for some time but then left to his native country (with a new stop in Berlin), and with two concerns: what happened to his precious Paris manuscript? And, how could he ensure minimum material comfort for himself and his fiancée? Times were really tough. ("My trip to Paris was terribly pointless", he confides in a letter to Boeck) and the situation alarming as he discovered the lack employment on returning to Norway. Getting a stable job was then much more complicated than it is nowadays and Abel and his friends really struggled. The former wrote more outstanding works, while the latter rushed to academic authorities in different countries. In Norway he

[3]. Letter from 24th October 1826, page 45 from the correspondence section of the Paper, published on the occasion of the centenary of his birth, online here (see figure 3).

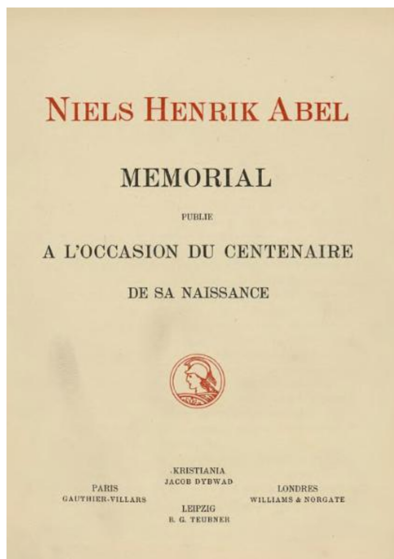
[4]. In fact, he would find it in June 1829 after Niels Abel's death but then the manuscript would be lost again another two times before an unexpected rediscovery in Florence (the missing last pages were found in 2002), where it remains today. For full details of such "negligent" management by the *Académie des Sciences*, you can read "Abel et l'Académie des Sciences" by René Taton, in *Revue d'histoire des sciences et de leurs applications*, 1948, volume 1, no. 1-4, pp. 356-358 (online in Persée).

was offered Abel a job to replace Professor Hansteen who was then travelling in Siberia. Crelle was thrilled to finally announce in a letter from 8th April 1829:

> *My dear, dear friend, I am now able to bring you some good news. The Ministry of Education has decided to summon you to Berlin, and offer you a job. (...) You can now contemplate the future without worries. (...) You will come to a truly good country, with a better climate, and you will be even closer to science and genuine friends by whom you are valued and loved.*

Unfortunately, Niels would never get to read this letter because he died two days earlier (then aged 27 years old) of tuberculosis (which was presumably contracted during his terrible stay in Paris) during a trip with his fiancée to Froland. As indicated by Holmoe in the preface to the edition of works of Abel from 1838:

> *Tirelessly he continued his scientific research, and it is possible that, especially in the last year of his life, he put in too much activity and effort, since he was a naturally unwell person with a weak and sensitive makeup.*



NIELS HENRIK ABEL

MEMORIAL

PUBLIÉ

A L'OCCASION DU CENTENAIRE

DE SA NAISSANCE

KRISTIANIA
JACOB DYBWAD

PARIS
GAUTHIER-VILLARS

LONDRES
WILLIAMS & NORGATE

LEIPZIG
B. G. TEUBNER

***Figure 3:* The memorial published to mark the centenary of Abel's birth** *(editions in Kristiania, Paris, London & Leipzig); such works were published in French at the time (1902).*

WORKS

If Niels Abel's works were slow in becoming fully recognised in France, they were nonetheless partially published during his lifetime, mainly in the famous Crelle's Journal (a much more practical title than the official German title *Journal für* *die reine und angewandte Mathematik, herausgegeben von Crelle*), at that time the collection of several (apparently) complete publications of works. As Christian Houzel[5] suggests, Abel's works can be classified into the following five categories:

1. Resolution (either possible or not) of algebraic equations by radicals: this is the subject of his first major piece of work with the impossibility for an equation of degree 5 (the theorem from now on known as the Abel-Ruffini theorem). His efforts would be completed by one of Galois's more general theories a few years later.

2. Study of transcendental functions: we know the number of functions defined as primitive from the functions with radicals (elliptic integrals) or reciprocal of the latter (elliptic functions). Abel studied them rigorously and in great detail and then classified them according to their properties. This was the main objective of the lost thesis of the *Académie des Sciences* in Paris but also the basis of a healthy rivalry with another promising mathematician, Jacobi (1804-1851).

3. Resolution of functional equations; in his works on transcendental functions, Abel encountered a problem regarding functional equations which he already knew from his mathematical youth. He provided some advances (though often unknown).

4. Integral transformations: these also emerged at some point as a tool in other studies but led to some specific publications.

5. Series (finite or infinite): just like Cauchy in his course at the *École polytechnique*, Abel incorporated the bases of this theory when he discovered the predominant lack of strictness. Here are his comments in a letter to Holmboe:

> *Even if we consider the simplest case, there is not in all mathematics a single infinite series whose sum has been rigorously determined. In other words, the most important areas of mathematics prove to be without a foundation.*

Consequently, he would go on to bring order and discipline to the methods employed: thus born were Abel's lemma, the Abel theorem, the Abel transform...

General Description of the Text

The text of this discussion, titled *Recherches sur la série binomiale* [Research

[5]. *The legacy of Niels Henrik Abel: the Abel bicentennial*, Oslo, Springer 2002.

on the Binomial Series], is the main part of Abel's work on the fifth category above; the original text was published in *Journal de Crelle* in 1826 (after a shorter text dealing with the same series), but its republication in France will be used as part of the complete works which were published in 1881 by the Norwegian State under the direction of the two prestigious mathematicians, Ludvig Sylow and Sophus Lie.

This article is divided into different parts that do not all present the same interest for us. The first is a short introduction to the entire article recalling some problems of the theory of numerical series. The second draws up a list of convergence theorems and their demonstrations. The third and fourth parts are a virtuoso work in the particular case of binomial series. Such deployed techniques then appear customary for the study of entire series; yet, at the time they impose on Abel the introduction of radius of convergence and the distinction between studies on open disc convergence and the edge of this disc. The fifth part exploits the results for binomial series in order to calculate the ensuing sum of other series.

SECTION I

In order to explain his approach, Abel provides a brief overview of the problems of series convergence, as identified in mathematical literature of the time; he was outraged of the little consideration given to such appropriate justifications - as if the manipulation of sums of an infinite number of terms was completely identical to that (which is more common) of the sums of a finite number of terms:

> *Usually the operations of the analysis are applied to infinite series in the same manner as if the series were complete, which does not seem to be without particular demonstration.*

Indeed, in order to be able to manipulate infinite sums (like the proposed examples of the product of two of these sums, of the composition of an infinite sum with a common function or the manipulation of divergent series), what must be proved first of all is the existence of such sums, that is to say, the convergence of the sequence of partial sums when the number of terms approaches infinity. This rigorous approach (mainly due to Cauchy) responds in fact to a century of hazardous manipulations of divergent series and to the controversies produced.

Then, Abel introduced the binomial series whose general term (indicated by *n*) is $m \times (m-1) \times ... \times (m-n+1) \times x^n / n!$. He recalls that there is only a finite number of nonzero terms if *m* is a natural integer and that the sum is therefore $(1+x)^m$ according to the binomial theorem of Newton. Ultimately, he intended to

> *Find the sum of the series*
>
> $$1+\frac{m}{1}.x+\frac{m(m-1)}{1.2}.x^2+\frac{m(m-1)(m-2)}{1.2.3}.x^3+etc.$$
>
> *for all real or imaginary values of x and m for which the series is convergent.*

This series had already been studied by Newton, Euler, Bolzano (whose paper is not even cited by Abel) and Cauchy (as well as the study of convergence and determination of the sum for a real argument). To be able to study this series, Abel had to create some reminders (from "the excellent work of M. Cauchy") and state some results (theorems I to VI) which form the pattern of this article's second section. Now to explain in detail such statements and their evidence with more contemporary notations: indeed, Abel used numerical series and then described the different behaviour of such series when a parameter was modified; nowadays it is simpler to speak directly of series of functions and to study the convergence and the properties of the sum as a function. Thus, we translate Abel's statements in terms of whole series even if this word is anachronistic compared to the text studied here.

COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

PAR M. AUGUSTIN-LOUIS CAUCHY,

Ingénieur des Ponts et Chaussées, Professeur d'Analyse à l'École polytechnique, Membre de l'Académie des sciences, Chevalier de la Légion d'honneur.

I.re PARTIE. ANALYSE ALGÉBRIQUE.

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Chez DEBURE frères, Libraires du Roi et de la Bibliothèque du Roi, rue Serpente, n.° 7.

1821.

Figure 4:* Augustin Cauchy (1789-1857) and his famous Cours d'analyse à l'École royale polytechnique *(1821).

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Abel begins by explaining the convergence of a numerical series: being given a real or complex sequence (ν_n), the series is considered with general term ν_n. Such a series is convergent if the sequence of partial sums (understand the sum of terms ν_k for *k* being from 0 to *n*) come together when *n* approaches infinity. It is otherwise considered divergent. Abel then recalls a necessary and sufficient condition of convergence for a series[6] (which from now on will be called the Cauchy criterion):

> *For a series to be convergent, it is necessary and sufficient that for ever increasing values of* m*, the sum* $\nu_m + \nu_{m+1} + \ldots + \nu_{m+n}$ *approaches zero indefinitely, regardless of the value of* n*.*

Then, as a final reminder, he explains in a sentence a necessary condition (but not sufficient) of convergence: if the series converges, then the general term approaches zero. Incidentally, he clarifies a somewhat original notation for readers of the twenty-first century:

> *represented in this paper by* ω *will be an amount which can be smaller than any given quantity*

In more conventional terms, a sequence will be smaller than ω if for every $\varepsilon > 0$, there is a value from which all terms of the sequence are bounded by ε. What Abel wrote, nonetheless, has not really been preserve beyond the nineteenth century; but it has the advantage of making it possible for him to translate the fact that a sequence approaches zero (and, in more general terms, to write the convergence of a sequence) in terms of inequalities which are easy to manipulate.

SECTION II

The first two theorems (numbered I and II) strictly state a result known today as Alembert's ratio test (providing, among other things, a way to calculate the radius of convergence of an infinite series). This result was already known before the work of Abel but was often used inappropriately, such as in the work of Bolzano on binomial series (*Binomischer Lehrsatz and Rein Analytischer Beweis*, 1817).

Let's consider a sequence (ρ_m) with positive terms so that the quotient ρ_{m+1}/ρ_m permits the finite limit α.

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[6]. The series here implicitly is of real or complex values: the argument to ensure that the Cauchy criterion is sufficient to the convergence is the completeness of the underlying space.

Theorem I. *If the limit α is strictly greater than 1, then the series with general term* ρ_m *is divergent and, more generally, any series with general term* $\varepsilon_m\rho_m$*, where* ε_m *does not approach 0, is divergent.*

Theorem II. *If the limit α is strictly less than 1, then the series with general term* ρ_m *is convergent and, more generally, any series with general term* $\varepsilon_m\rho_m$*, where* ε_m *is limited,*[7] *is convergent.*

The proof of these theorems is based on the comparison of the series with general term ρ_m with the geometric series with the value α', with α' being strictly between α and 1, that is to say, the comparison with the series with general term $(\alpha')^m$. As we know explicitly the expression of partial sums of the geometric series (and more particularly its Cauchy slices (that is to say, the sums of the terms of the series for the evidence included between fixed values), we know that this one converges if (and only if) its value is of a modulus which is strictly less than 1.

In his evidence, Abel is a little less precise: he seems to assume that from a certain value, $\rho_{m+1} < \alpha \rho_m$, that is to say, that after the sequence ρ_{m+1}/ρ_m approaches α by lower and distinct values of α (the terms of the sequence approach α while remaining strictly less than α); furthermore, the statement implies that $\alpha \neq 1$. Nevertheless, it is extremely important to acknowledge that these points are only details that do not actually fundamentally change the nature of the result and that repeating such results would serve to polish these details which are written down.

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Abel then continues with a more technical theorem:

Theorem III. *In which* (t_m) *is a real sequence,* (p_m) *is the sequence of partial sums of the series with general term* t_m *and* (ε_m) *is a positive decreasing sequence. If the sequence* (p_m) *is raised by* δ*, then the partial sums of the series with general term* $\varepsilon_m t_m$ *is raised by* $\delta\varepsilon_0$*.*

This result is extremely importance in all the evidence that follows: for it shows how a series can be controlled by decomposing its general term as the product of a

7. Abel adds by 1, but it is clear that this choice of constant does not occur in his evidence of this theorem.

positive decreasing term and of a term whose sums are monitored. The demonstration is based on the technical trick which entails matching t_m with the difference p_m - p_{m-1} in order to get a sum that involves firstly p_m and secondly *(ε_m - ε_{m-1}):* this great idea makes it possible to shift the difficulties to be reduced to terms upon which are the hypotheses of the theorem. The manipulation implemented, which imitates the integration by parts by replacing the functions for sequences and the integrals for sums, will now called the Abel transform (or summation in parts).

Example of the Application of the Abel Theorem

The Abel theorem can be applied to many studies of convergence. A primary example relates to the elementary case of "alternating" series, that is to say, of its coefficients $(-1)^n\varepsilon_n$ with (ε_n) decreasing towards 0: then $t_n = (-1)^n$ is taken and it is verified that the partial sums of this series are bounded by 1. Consequently, the alternating series is convergent. This result, which was already known as Leibniz's, was referred to as the special criteria of alternating series. Thus the preceding example can be generalised with $t_n = e^{in\theta}$ for θ, which is not a whole multiple of 2π. The character which is bounded by the partial sums then comes from the explicit calculation of the geometric sums.

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The following two theorems relate to the continuity[8] of a sum of a functional series, firstly in the case of infinite series and then in the case of infinite series that depend on a parameter.

Theorem IV. *In which f is the sum of the infinite series of coefficients* (ν_m)*. Let's suppose that this infinite series is convergent for the argument δ > 0 (that is to say, that its radius of convergence is greater or is equal to δ). Then:*

➔ *the infinite series converges for any argument with absolute value less than δ;*

➔ *the function f is continuous to the left on [0, δ].*

In the essay proposed here on theorem IV, the two different points of Abel's statement have been highlighted. The first corresponds to a definition of what the

8. The definition of continuity which Abel recalls is really more about the continuity on the left.

radius of convergence is called today as the upper limit of the total reals for which the infinite series converges; indeed, the first point indicates that for any argument which is strictly less (in absolute value) than the radius, the series is convergent. The second point is to do with an even finer result and indicates the property of continuity of the sum of the infinite series on the area of convergence, including the edge of the circle if the series converges there. The proof involves cutting the defining series so as to exploit the hypotheses of convergence (via the Abel transform, theorem III) and the continuity of partial sums (which are polynomials).

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The following theorem is designed to study the continuity related to a parameter in the coefficients of the infinite series.

Theorem V. *In which* (v_n) *is a result of continuous functions on* [a,b]*. Let's suppose that the infinite series in which the coefficients of coefficients* (v_n) *are, for any* $x \in$ [a,b]*, convergent for the argument* $\delta > 0$*. Then, for any* $\alpha \in [0, \delta]$*, the function*

$$f : x \mapsto \sum_n v_n(x)\alpha^n$$

is continuous on [a,b].

This result, which is a bit more technical because of the two variables (those of the functions v_n, denoted by *x* and that of the infinite series, denoted by α), explores in fact the notion of uniform convergence even if this is not stated so. Abel, indeed, realised that the convergence of the series at one point is not enough to convey the property of continuity on a vicinity of this point. It is much easier to say that Abel resumed Cauchy and his course in 1821 as a footnote, in which he shows a counterexample with the trigonometric series which has the general term $x \mapsto \frac{1}{k}\sin(kx)$ (which is not continuous to the multiple blundering arguments of π).

The Notion of Uniform Convergence

Uniform convergence is a much stronger condition than simple convergence.

➔ In the case of simple convergence, for any threshold ε > 0 and any element *x* of the considered domain, a value can be found from which the distance $|f_n(x)-f(x)|$ becomes less than ε. A priori, if $y \neq x$ is chosen, then the value *N* whose distance $|f_n(x)-f(x)|$ is less than ε, will be different and there will not necessary be a value which could be suitable for all the arguments *x*.
➔ In the case of uniform convergence, a value can be found from which the distance $|f_n(x)-f(x)|$ becomes less than ε for any *x*. This condition is therefore much stronger: a sequence of functions which converges uniformly altogether, converges simply on this one (and for each x it is enough to consider the value which is common to all the arguments).

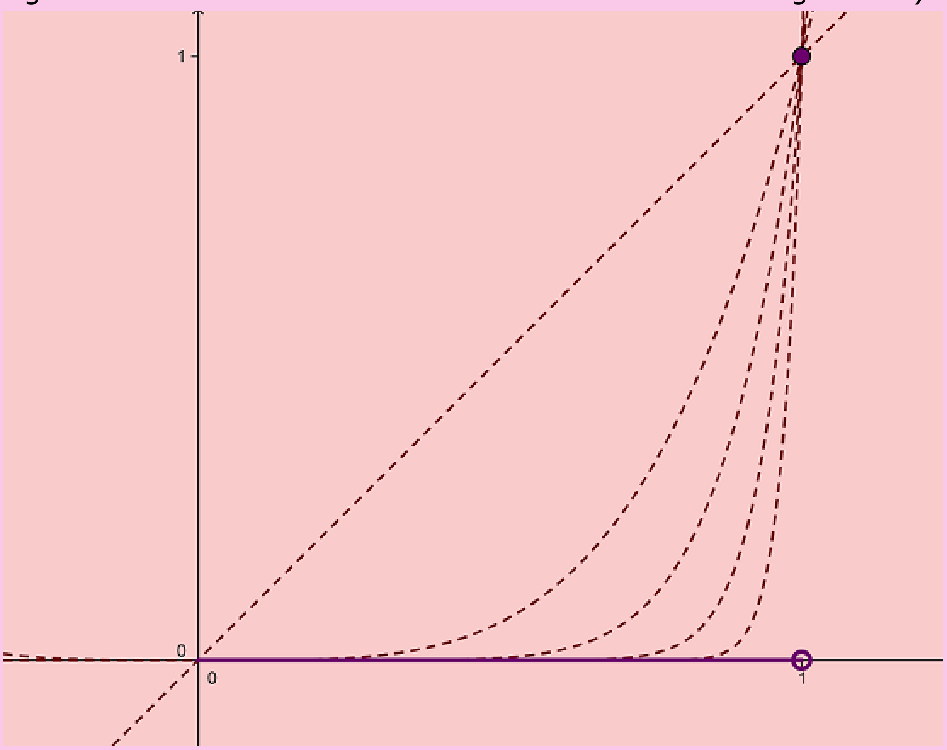

***Figure 5:* Non-uniform convergence**. *The functions $f_n(x)= x^n$ (whose curves are represented by the dotted lines) converge on the segment [0,1] around the discontinuous function (of value 0 everywhere except on 1 where it is worth 1) which is represented by the violet curve. This convergence is not uniform on the segment [0,1] but it is on any segment of [0,a] where a < 1.*

In this diagram, the evidence resembles that of the theorem IV with a cutting via isolating a polynomial function and via the increasing of the rest by a geometric sequence whose value is strictly less than 1, multiplied by a function θ. This makes it possible to obtain easily an increase in the remainder for a fixed *x*; unfortunately, such an increase depends on *x* and the sequence of Abel's argument is actually wrong (just like the case with Cauchy but this time the affirmation demonstrated is correct) because he used the same increase on an interval between ($x - \beta$) and x.

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The last theorem of this section no longer concerns the regularity of functions but a role of manipulation for the product of series (now called the Cauchy product).

Theorem VI. *In which two convergent numerical series with general term* (ν_n) *and* (ν'_n), *so that the numerical series with general term* $\rho_n = |\nu_n|$ *and* $\rho'_n = |\nu'_n|$ *is also convergent. The series with general term* $r_n = \sum_{k=0}^{n} \nu_k \nu'_{n-k}$ *is thus convergent whose sum verifies*

$$\sum_n r_n = \left(\sum_n \nu_n\right)\left(\sum_n \nu'_n\right).$$

Or, in somewhat more contemporary terms, the Cauchy product of two absolutely convergent series still defines a convergent series. The evidence consists of recognising in the partial sum of r_n the product of partial sums ν_n and ν'_n, and in addition increasing the remaining terms through the use of absolute convergence.

No sooner had he made this demonstration, had Abel engaged with a result in which the statement is not stated as a theorem as such.

In which two numerical convergent series with general term t_n and t'_n, so that the series with general term $\sum_{k=0}^{n} t_k t'_{n-k}$ is convergent.

Thus

$$\sum_n \left(\sum_{k=0}^{n} t_k t'_{n-k}\right) = \left(\sum_n t_n\right)\left(\sum_n t'_n\right).$$

The evidence of this result combines the different theorems of this section:

- Introducing the infinite series (whose coefficients are the t_n and t_n' respectively) taken into $\alpha \in [0,1[$, which are absolutely convergent according to theorem II
- Calculating the product of these series with theorem VI.
- Finally, making the transition to the limit $\alpha \to 1$ thanks to the continuity obtained

through theorem IV.

SECTION III

The objective of this fairly extensive section is to study the convergence of the binomial series in the case of an argument *x* of different modulus of 1, and to calculate when the sum of the series exists. Abel thus proceeded in several steps. He rewrote all the quantities according to real variables. Here is a brief summary of the notations for the reader who wishes to follow the calculations in their entirety:

	écriture cartésienne	*écriture géométrique*
m	$k+ik'$	
x	$a+ib$	$\alpha e^{i\phi}$
$\frac{m-\mu+1}{\mu}$		$\delta_\mu e^{i\gamma_\mu}$
$\frac{m(m-1)\dots(m-\mu+1)}{\mu(\mu-1)\dots 1}x^\mu$		$\lambda_\mu \alpha^\mu e^{i\theta_\mu}$
$(m)=\sum_{\mu=0}^{\infty}\frac{m(m-1)\dots(m-\mu+1)}{\mu(\mu-1)\dots 1}x^\mu$	$p+iq$	$f(k,k')e^{i\psi(k,k')}$

➔ The first step consists of identifying the real and imaginary parts (*p* and *q*) of the desired sum *φ(m)* in accordance with these different parameters: the formula numbered 2 in Abel's text (on page 74) is obtained. In order to determine the convergence or divergence in the case $\sqrt{a^2+b^2}=\alpha\neq 1$, it is therefore enough to apply the D'Alembert's principle (theorems I and II of section 2).

➔ Once the convergence was established for $\alpha < 1$, Abel used theorem VI on the series products in order to show that the sum verifies the functional equation *φ(m+n) = φ(m) φ(n)* (formula 3).

➔ The following step consists of solving this equation in stages:

- A primary change of an unknown function (an auxiliary function is solved with *θ*)
- A recurrence in order to obtain integer values (formula 8)
- The transition to rational values (formula 9)
- Then, with a continuity argument from theorem B, the transition to real values.

➔ Formulas 11 to 16 describe the steps for returning from this result on the auxiliary function θ to the desired sum φ through the modulus of this last function.

Section IV

This section aims to clarify the borderline case of the arguments *x* of modulus 1. Accordingly, the discussion will focus on the real part of the exponent *m*.

➔ First case: If the real part *k* of *m* is less than or equal to −1, the series general term does not approach 0 because its module λ_μ does not approach 0: there is trivial divergence.

➔ Second case: If the real part *k* of *m* is positive, then the series with general termλ_μ is convergent according to the Cauchy criterion[9]. Subsequently, the use of the Abel transform (theorem II) demonstrates that the binomial series converges.

➔ Third case: If the real part *k* of *m* belongs to] −1,0[,then it is multiplied by *x+1* and we are thus reduced to the preceding case which establishes the convergence.

The section is finally devoted (on the one hand) to a reminder of the established results, of convergence and the value of the sum, and (on the other hand) to some particular cases: or, more precisely, A: case where *m* is a real number; B: case where *x* is a real number; C: case where *m* and *x* are all real; D: case |*x*|= *1* ; E: case *m* real and |*x*|= *1* , F: case where *x* is purely imaginary) where the formula of the sum is simplified in order to find previously known results.

Section Vs

This last part makes it possible

through appropriate transformations of the previous expressions [to] *still deduce several others, among which are some very remarkable ones* .

What is noticed (among other things) is the development in infinite series of

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[9]9. Recall the Cauchy criterion for a series: for any threshold ε > 0, exists a value from which all sums of a finite number of terms of consecutive indices are increased (in modulus) by ε. The completeness of the body of real or complex ones ensures a series which verifies this property will converge (property of completeness of such bodies).

the functions *ln* and *arctan* (arctan $\alpha = \alpha - 1/3\ \alpha^3 + 1/5\alpha^5 + \ldots$) as well as other admirable identities among trigonometric functions.

CONCLUSION

This article written by Abel is officially a virtuoso work for studying a very particular series but one in which there is, in addition to the rare rigor of that time (not including the error in the evidence of theorem V), some general tools for the study of numerical series. Taking on the ideas and work of Cauchy regarding convergence (particularly, the study of convergence via the Cauchy criterion), Abel introduces calculation methods which adapt for numerous semi-convergent series and which prefigures (among other things) the uniform convergence of Karl Weierstrass ("Sur la théorie des séries entières" ['On the Theory of Infinite Series], written in 1841 but published only in 1894) and Louis-Augustin Cauchy ("Notes sur les séries dont les divers termes sont des fonctions continues d'une variable réelle ou imaginaire, entre des limites données" [Notes on Series whose Various Terms are Continuous Functions of a Real or Imagined Variable among Given Limits] published in *Comptes Rendus de l'académie des sciences* (Paris, 1853), the text in which he acknowledges his mistake from 1821, which was highlighted by Abel, before introducing the Uniformly Cauchy Sequence for the series of functions).

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(July 2011)

(Translated by John Moran, published December 2014)