To an unknown mathematician!

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It is said in the New Testament that the apostle Paul, when addressing the Athenians, complimented them for having inscribed “To an unknown god” onto one of their altars. Reading the Babylonian tablet YBC 7289 (*Yale Babylonian Collection*), one might well be tempted to erect another altar to this unknown mathematician who, nearly four thousand years ago, engraved into clay what is for us one of the most extraordinary mathematical documents that exists. The YBC 7289 tablet was written between 1900 and 1600 BCE. Its round shape and size (about 8 cm in diameter) make it easy to handle, which indicates that it is probably the work of an apprentice.

An object which, in its time, was perhaps a mere “student’s copy” has thus for us become a major testimony to how the Babylonians perceived mathematics. As no treaty on this subject has apparently survived from the period, it is difficult, if not impossible, to recreate the context in which mathematical study took place.

Among the mathematical clay tablets we possess, many are of an educational nature, setting out the terms of a problem followed by its solution. Rare are tablets such as the YBC 7289, which, in such a disproportionately small space, provide so much food for thought on how the Babylonians envisaged the abstract notion of number.

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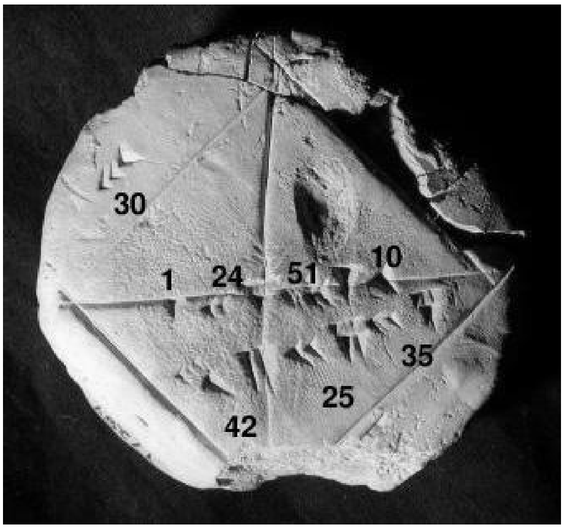
In modern terminology, the YBC 7289 tablet makes a connection between a geometric object (a square and its diagonals) and an algebraic object, which is the square root of 2, expressed as $\sqrt{2}$, the positive number which, when multiplied by itself, comes to 2.



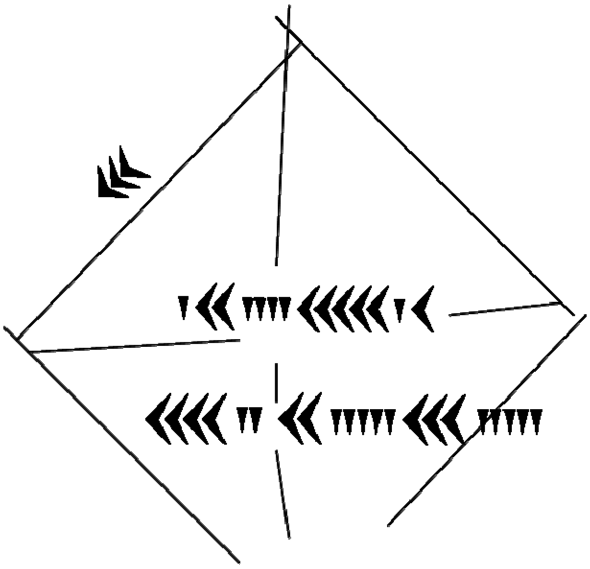
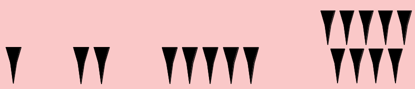


Figure 1 : ***The YBC 7289 tablet, with a schematic representation to the right.*** *We can make out a square and its two diagonals. The side of the square is shown as being of length 30; the length of the diagonal is shown by the group of chevrons and wedges signifying 1 24 51 10.*

The tablet does not set out a mathematical problem. All in all it displays a square, its two diagonals, and a few markings, usually described as "wedges" or "chevrons". These markings correspond to values written in the Babylonian number system (see boxed text).

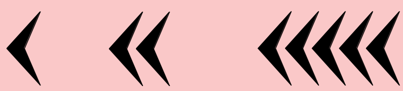
Babylonian calculation

The number system used in the YBC 7289 contains only two symbols: wedges and chevrons. To write the numbers 1 to 9, the number of wedges corresponds to the given value:



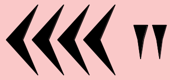
1 2 5 9

The value 10 is indicated by a chevron. To write twenty, thirty, forty or fifty, either two, three, four or five chevrons are aligned side by side:

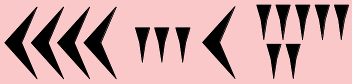


10 20 50

To write a number such as 43, all that is needed is to align the corresponding number of chevrons and wedges, as below:



To write 60, another single wedge is used. Two wedges thus signify 120 (= 2 x 60), three wedges 180 (= 3 x 60), and so on up to 9 wedges for 540 (= 9 x 60). To write the value 600, again another single wedge is used, and so on.[1] For example, in the Babylonian number system, the number 2597 [= (4 x 600) + (3 x 60) + (1 x 10) + 7 = 43 x 60 + 17] is written as follows:



The Babylonian system does not use zero or the decimal point. Context is therefore needed to determine the value of a given marking: a single wedge may very well have a value of 1, 60, or 3600 (= 60^2), etc., or, equally, a value of 1/60, 1/3600, etc. In particular, the three wedges in the top left of the YBC 7289 tablet can in theory either designate 30 or 1/2 (=30/60), 1800 (=30 x 60), etc. Nowadays the most common interpretation is that these three chevrons have a value of 30 (even if Eleanor Robson and David Fowler have pointed out that the correct value may possibly be ½). The number at the very bottom has a value of 42.426389…[2] (the transition from root sixty to root ten makes it impossible to give an exact decimal number), and the middle number a value of 1.4142129…[3] (ditto).

A simple calculation shows that the product of 30 by 1.4142129… is equal to 42.426389…. The value 1.4142129… corresponds to what we express as √2, the square root of 2.

The tablet may give the answer to the following problem: "Calculate the length of the diagonals of a square with sides of a length of 30." The student has sketched a solution, copying the problem's only datum and multiplying it by 1.4142129… to obtain 42.426389…, the answer to the exercise.

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1. Given that up to nine wedges and five chevrons may be aligned in succession, the transitions occur at 10 (one chevron), 60 (one wedge), 600 (one chevron), 3600 (one wedge), 36 000 (one chevron), 216 000 (one wedge), etc. The addition of wedges thus occurs at 1, 60, 3600 (i.e. 60^2), 216 000 (i.e. 60^3), etc. The system is sexagesimal (base 60, to be compared with the decimal system, base 10, where the addition of a new column in the number occurs at 1, 10, 10^2, 10^3), and uses two symbols, the wedge (up to but not including ten) and the chevron (up to but not including six).
2. The representation of 42 25 35 can in effect be interpreted as $42 + 25/60 + 35/60^2$, i.e. 42.42638888…
3. The representation of 1 24 51 10 can in effect be interpreted as $1 + 24/60^2 + 51/60^3 + 10/60^3$ i.e. 30 547/21 600 = 1.41421296 (root of 2 to the nearest 0.5×10^{-6}).

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It is the presence of this value 1.4142129…, for us the square root of 2, which makes this tablet so remarkable. Its status is very different from that of the other values written down by the scribe, which were probably chosen to assign simple data. To write the value of 1.4142129…, the scribe had to know that this was the value by which to multiply the length of the side to obtain the length of the diagonal, regardless of the length chosen for the side of the square. In other words, he had to know that the square root of 2 is a fundamental constant of geometry, in the same way as the number Π (which gives the relation of the circumference of a circle to its diameter[4]). While remaining cautious of statements of this kind, we may conclude that in YBC 7289 the square root of 2 should not be taken as the result of a measurement or calculation, but as an object of such a highly abstract nature that the tablet contains arguably the first example of "number" in the fullest and most theoretical sense that mathematicians now attribute to the term.

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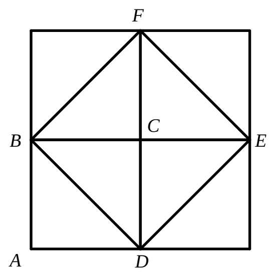
The tablet provides an elementary geometrical demonstration of the fact that the diagonal and the side of the square equal the modern square root of 2, as can be seen in the following figure:



If we suppose that the *ABCD* square is 1, its area is equal to 1. The *BDEF* square contains four triangles, i.e. twice more than *ABCD* (which contains only two). *BDEF*'s area is thus 2, and its *BD* side thus measures √2. It therefore follows that the relationship between BD/AB – that of the diagonal to the side of the *ABCD* square – is √2.

4. It should be noted that we possess little evidence with which to speculate about the Babylonians' knowledge of the number π.

We do not know which methods the Babylonians used to come to the conclusion that, in a square, the diagonal is √2 longer than the side. To make an approximate calculation of the square root of 2, and of all square roots for that matter, it seems they were aware of what we now know as Heron's method (or at least a variant of it). To make an approximate calculation of the square root of a number *a*, this method involves positing an initial approximation a_0 (for example, the closest integer to √a) and, then, for every integer $n \geq 1$, to calculate the values a_1, a_2, a_3, etc., obtained by the formula

$$a_n = (a_{n-1}+a/a_{n-1})/2.$$

Taking the value 1 as an initial approximation of √2, the method produces successively the values 3/2, 17/12, 577/408, then 665857/470832. The value written on YBC 7289 is precise to the nearest millionth; its degree of precision is intermediate between that of 577/408 and 665857/470832. Written in the Babylonian number system, the approximation of √2 that we find on the YBC 7289 is the most accurate possible for a calculation precise to three sexagesimal rows in its fractional part – in base 60, the equivalent of three figures after the decimal point.

Calculating a square root using the method of Heron of Alexandria (first century AD)

Even today, this is still the most effective known method for calculating the square root of a number *a*. Starting from an initial value a_0 (which we can take as equal to *a*), the method consists in calculating the terms a_1, a_2,… defined by the formula:

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$$

To demonstrate that the a_n do indeed converge with $\sqrt{a}$, the idea is to establish that $(a_n)_n$ is a decreasing and lower-bound – and therefore converging – series, and that its only possible limit is $\sqrt{a}$.

The fact that $(a_n)_n$ is decreasing manifests itself at two stages. First, we establish that each *n* from 1 on is greater than $\sqrt{a}$. This result is the consequence of the following equality:

$$a_{n+1} - \sqrt{a} = \frac{1}{2a_n}\left(a_n^2 + a - 2a_n\sqrt{a}\right) = \frac{1}{2a_n}(a_n - \sqrt{a})^2 > 0$$

The second stage consists in calculating the difference between two consecutive terms of the series:

$$a_{n+1} - a_n = \frac{1}{2a_n}\left(a_n^2 + a\right) - a_n = \frac{1}{2a_n}(a - a_n^2)$$

Since we know that $a_n > \sqrt{a}$, the expression $a{-}a_n^2$ is a negative number, so a_{n+1} is indeed lower than a_n, in other words, the $(a_n)_n$ series is decreasing. As the series is lower bound (by 0), it is convergent, and its limit L must verify the equality of $L = \frac{1}{2}\left(L + \frac{a}{L}\right)$, that is $L^2 = a$, and therefore $L = \sqrt{a}$ (since $L > 0$).

The rate of convergence of the a_n towards $\sqrt{a}$ is known as "quadratic": in short, the number of exact decimals doubles at each new term in the series. This can be seen in the relationship obtained above:

$$a_{n+1} - \sqrt{a} = \frac{1}{2a_n}(a_n - \sqrt{a})^2$$

When n is a large number, a_n approaches $\sqrt{a}$. This gives the following approximate relationship:

$$a_{n+1} - \sqrt{a} \approx \frac{1}{2\sqrt{a}}(a_n - \sqrt{a})^2$$

Note d_n, the difference between a_n and $\sqrt{a}$ (that is, the difference $a_n - \sqrt{a}$. The previous relationship indicates that d_{n+1} is proportional to the square of d_n, hence the expression "quadratic convergence". For example, if the degree of precision attained by a_n in approximating $\sqrt{a}$ is precise to a thousandth (i.e. d_n to the order of 10^{-3}, then that attained by a_{n+1} is precise to a millionth (d_{n+1} to the order of d_n^2, that is $(10^{-3})^2 = 10^{-6}$).

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It is difficult to assess the possible developments subsequently undertaken by the Babylonians in this area. While many hypotheses can be advanced, ranging from the search for the ultimate decimal point of √2 (which, as we now know, does not exist) to questions about the fundamental nature of the square root of 2 (the fact that it is an irrational number that does result from the division of one integer by another), to the most advanced study of the connections between geometry and algebra, most of these hypotheses are hampered by a lack of evidence from the period.

From the sixth and fifth centuries BC onwards, it was the Greeks who would explore these questions further and lay down the geometrical foundations of the theory of numbers. Playing their part in this story are philosophers like Plato and Aristotle, and before them Euclid, Theodorus and Archytas – all of them true mathematicians who would inject such momentum into geometry and the theory of numbers that, even today, we still acknowledge them as our forebears. The YBC 7282 tablet nevertheless demonstrates that, well before this, the Babylonians already had a highly detailed understanding of some of the connections between algebra and geometry, an understanding that cannot be reduced to a simple matter of calculation techniques, however sophisticated they may have been.

Looked at in a (necessarily) cautious light, the author of the YBC 7289 – or his teacher – is perhaps our oldest authentic mathematician colleague of whom a record survives.

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